University of Global Village (UGV)

Course Title: **Complex Variable, Vector Analysis and Coordinate Geometry**
Course Code: 0541-2201
Credit: 3
CIE: 90 marks
SEE: 60 Marks
Exam hour: 3

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Course Outlines

Course Code: Math 0541-2201 Semester End Exam (SEE): 3 Hours	Credit: 03 CIE Marks: 90 SEE marks: 60
Course Learning Outcomes (CLO): After successful completion of the course students will be able to -	
CLO1 Define the basic terminology and theorems associated with Complex variable, Vector Analysis and Coordinate Geometry.	
CLO2 Properties of Complex number, Continuity and Differentiability of Complex number, Analytic functions, Residues, Cauchy integral formula, Harmonic Functions and Applications of complex numbers in Engineering.	
CLO3 Describe Two Dimension Geometry, Three-dimension Geometry, Plane, Cube.	
CLO4 Operation of vector Analysis, vector differentiation and Integrations, Different applications of vector Analysis.	
CLO5 Apply the acquired concepts of Complex variable, Vector Analysis and Coordinate Geometry in engineering.	

Course Content Summary

SL.	Content of Courses	Hrs	CLO's
1	Definition of complex number, Notation, Modulus and argument of complex number, Real and Imaginary part of Complex number, Polar Form of Complex Number, Related Mathematics. Geometrical Interpretation of Complex Number, Graphical Representation of complex equations, Circle and Ellipse related Problems	8	CLO1, CLO2
2	General functions of complex Number, Limits of functions of complex variables and related theorems, Continuity of functions of complex variables. Definition Analytic Function, Entire Function, Singularity, and related problems, Cauchy's Integral theorem and related Conjugate, Laurent's Theorem, Residue, Cauchy Residue Theorem, CRT Related mathematics.	6	CLO1, CLO5

3	Introduction, Change of axes, Transformation of co- ordinates, Pair of straight lines, General equation of second degree, Circle, Equation of circle and mathematics, Conic section: Parabola, Ellipse, Hyperbola and properties, The general equation of second degree, Equation of Conics and Properties of Conics, Direction cosines and direction ratios and related Mathematics, Straight Line, Plane, Equation of Plane and Related mathematics, The shortest distance, Equation of Shortest distance and Related mathematics.	10	CLO3, CLO5
4	Definition of Vectors, Scalars and vectors, Elementary operations of vectors, Dot product and Vector Product of vectors, Multiplication of vectors by scalars, Angle Between two vectors and Geometrical Representation of vectors, Spherical and Cylindrical systems, Divergence and Curl, Gradient, Physical significance of Gradient, Divergence and Curl, Gauss's Theorem and its application, Gauss's Theorem and its application, Green's theorem with applications, Stroke's theorem and its application.	10	CLO4, CLO5

Course Plan Specifying content, CLO's, Teaching Learning, and Assessment strategy mapping with CLO's

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
1	Complex Number • Definition of complex number • Modulus and argument of complex number • Real and Imaginary part of Complex number	Lecture, Discussion	Quiz	CLO1

	• Polar Form of Complex Number • Related Mathematics			
2	• Geometrical Interpretation of Complex Number • Graphical Representation of complex equations • Circle and Ellipse related Problems	Discussion, Oral Presentation	Written Assignment	CLO1
3	General functions of complex Number • Limits • continuity of functions of complex variables. • Analytic Function • Entire Function • Singularity • and related problems	Oral Presentation	Oral Presentation	CLO1
4	Cauchy's Integral theorem • theorem • and related problems	Group Work	Group Assignment	CLO1
5	Harmonic and Conjugate harmonic • Definition of harmonics. • Properties of harmonic functions • Cauchy Reimann PDE, Conjugate	Discussion	Presentation	CLO2

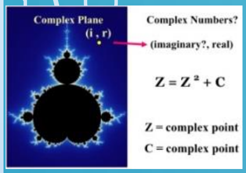
6	• Finding harmonic Conjugate • Related Mathematics, Solution of these equations together with applications.	Group work	Quiz, Written Assignment	CLO2
7	Residue Theorem and Related mathematics • Residues, Cauchy's Residue Theorem • examples • Laurent's Theorem • Cauchy	Lecture, Discussion	Oral Presentation, Quiz	CLO1
8	2-D Geometry • Introduction • Change of axes • Transformation of co-ordinates • Pair of straight lines	Discussion, Oral Presentation	Group Assignment, Quiz	CLO3
9	General equation of second degree • Circle, Equation of circle • Mathematics • Tangents and Normal • related Mathematics.	Oral Presentation	Presentation, Written Assignment	CLO2
10	Conic section: • Parabola • Ellipse • Hyperbola, and properties.	Oral Presentation	Quiz, Presentation	CLO3

11	Co-ordinate system • Direction cosines • direction ratios • related Mathematics • Straight Line, Plane, Equation of Plane • and Related mathematics	Group Work	Written Assignment, Oral Presentation	CLO3
12	The shortest distance • Equation of Shortest distance • Conditions of lines • and Related Mathematics • Cube and related Mathematics.	Discussion, Oral Presentation	Group Assignment, Presentation	CLO3
13	Vector • Definition of Vectors • Scalars and vectors • Equality of vectors • Elementary operations of vectors.	Discussion, Oral Presentation	Quiz, Group Assignment	CLO4
14	• Dot product and Vector Product of vectors • Multiplication of vectors by scalars • Angle Between two vectors	Oral Presentation	Written Assignment, Quiz	CLO4

	• and Geometrical Representation of vectors			
15	Vector Differentiation • Divergence • Curl • Gradient • Physical significance of Gradient, Divergence and Curl • Related theorems and mathematics	Lecture, Discussion	Oral Presentation, Group Assignment	CLO4
16	• Vector Integration • Position vector • Green's theorem with applications	Practical Work	Presentation, Quiz	CLO4
17	• Gauss's Theorem and its application • Stroke's theorem, and its application.	Reading Assignment	Quiz, Written Assignment, Oral Presentation	CLO4

Week 1
Topics: Complex number
Page no (7-40)

COMPLEX NUMBERS



You can use both real and imaginary numbers to solve equations

At GCSE level you met the Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The part under the square root sign is known as the 'discriminant', and can be used to determine how many solutions the equation has:

$$b^2 - 4ac > 0 \quad \longrightarrow \ 2 \; real \; roots$$

$$b^2 - 4ac = 0 \quad \longrightarrow 1 \; real \; root$$

$$b^2 - 4ac < 0 \quad \longrightarrow 0 \; real \; roots$$

The problem is that we cannot square root a negative number, hence the lack of real roots in the 3rd case above

To solve these equations, we can use the imaginary number 'i'

$$i = \sqrt{-1}$$

The imaginary number 'i' can be combined with real numbers to create 'complex numbers'

An example of a complex number would be:

$$5 + 2i$$

Complex numbers can be added, subtracted, multiplied and divided in the same way you would with an algebraic expression

1A

You can use both real and imaginary numbers to solve equations

1) Write √-36 in terms of i

This sign means the **positive** square root

$\sqrt{-36}$

Split up using surd manipulation

$\sqrt{36}\sqrt{-1}$

Simplify each part → √-1 = i

$= 6i$

2)Write √-28 in terms of i

$\sqrt{-28}$

Split up into a positive and negative part

$\sqrt{28}\sqrt{-1}$

Split up the 28 further…

$\sqrt{4}\sqrt{7}\sqrt{-1}$

Simplify each part

$= 2\sqrt{7}i$

This is usually written in this way

$= 2i\sqrt{7}$

COMPLEX NUMBERS

You can use both real and imaginary numbers to solve equations

$x^2 + 9 = 0$
Solve the equation:

$$x^2 + 9 = 0$$

Subtract 9

$$x^2 = -9$$

Square root – we need to consider both positive and negative as we are solving an equation

$$x = \pm\sqrt{-9}$$

Split up

$$x = \pm\sqrt{9}\sqrt{-1}$$

Write in terms of i

$$x = \pm 3i$$

You should ensure you write full workings – once you have had a lot of practice you can do more in your head!

COMPLEX NUMBERS

You can use both real and imaginary numbers to solve equations

$x^2 + 6x + 25 = 0$

Solve the equation:

→You can use one of two methods for this

→Either 'Completing the square' or the Quadratic formula

$(x+3)^2$

$(x+3)(x+3)$

$x^2 + 6x + 9$

Imagine squaring the bracket

This is the answer we get

The squared bracket gives us both the x^2 term and the 6x term
→ It only gives us a number of 9, whereas we need 25 – add 16 on!

Completing the square

$x^2 + 6x + 25 = 0$

Write a squared bracket, with the number inside being half the x-coefficient

$(x+3)^2 + 16 = 0$

Subtract 16

$(x+3)^2 = -16$

Square root

$x + 3 = \pm\sqrt{-16}$

Subtract 3

$x = -3 \pm \sqrt{-16}$

Split the root up

$x = -3 \pm \sqrt{16}\sqrt{-1}$

Simplify

$x = -3 \pm 4i$

If the x term is even, and there is only a single x^2, then completing the square will probably be the quickest method!

COMPLEX NUMBERS

You can use both real and imaginary numbers to solve equations

$x^2 + 6x + 25 = 0$

Solve the equation:

→You can use one of two methods for this

a = 1

b = 6

c = 25

→Either 'Completing the square' or the Quadratic formula

The Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sub in values

$$x = \frac{-6 \pm \sqrt{(6)^2 - (4 \times 1 \times 25)}}{2(1)}$$

Calculate the part under the root sign

$$x = \frac{-6 \pm \sqrt{-64}}{2}$$

Split it up

$$x = \frac{-6 \pm \sqrt{64}\sqrt{-1}}{2}$$

Simplify the roots

$$x = \frac{-6 \pm 8i}{2}$$

Divide all by 2

$$x = -3 \pm 4i$$

If the x^2 coefficient is greater than 1, or the x term is odd, the Quadratic formula will probably be the easiest method!

COMPLEX NUMBERS

You can use both real and imaginary numbers to solve equations

Simplify each of the following, giving your answers in the form: $a + bi$

$a \in R \text{ and } b \in R$

where:

This means a and b are real numbers

1) $(2+5i)+(7+3i)$

$= 9+8i$

Group terms together

2) $(2-5i)-(5-11i)$

$= 2-5i-5+11i$

$= -3+6i$

'Multiply out' the bracket

Group terms

3) $6(1+3i)$

$= 6+18i$

Multiply out the bracket

1A

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra.

$$i = \sqrt{-1}$$

You can also use the following rule to simplify powers of i:

$$i^2 = -1$$

Multiply out the following bracket

$(2 + 3i)(4 + 5i)$

Multiply put like you would algebraically (eg) grid method, FOIL, smiley face etc)

$= 8 + 12i + 10i + 15i^2$

Group i terms, write i^2 as -1

$= 8 + 22i + 15(-1)$

Simplify

$= -7 + 22i$

1B

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra.

You can also use the following rule to simplify powers of i:

$$i = \sqrt{-1}$$

$$i^2 = -1$$

<u>Express the following in the form a + bi</u>

$(7-4i)^2$

Write as a double bracket

$= (7-4i)(7-4i)$

Multiply out

$= 49 - 28i - 28i + 16i^2$

Group i terms, write i^2 as -1

$= 49 - 56i + 16(-1)$

Simplify

$= 33 - 56i$

1B

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra.

You can also use the following rule to simplify powers of i:

$$i=\sqrt{-1}$$

$$i^2=-1$$

Simplify the following:

$(2-3i)(4-5i)(1+3i)$

Start with the first 2 brackets

$(2-3i)(4-5i)$

Multiply out

$=8-12i-10i+15i^2$

Group i terms, replace i^2 with -1

$=8-22i+15(-1)$

Simplify

$=-7-22i$

Now multiply this by the 3rd bracket

$(-7-22i)(1+3i)$

Multiply out the brackets

$=-7-22i-21i-66i^2$

Group i terms and replace i^2 with -1

$=-7-43i-66(-1)$

Simplify

$=59-43i$

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra.

You can also use the following rule to simplify powers of i:

$$i = \sqrt{-1}$$

$$i^2 = -1$$

Simplify:

1) i^3

$= i^2 \times i$

$= -1 \times i$

$= -i$

Split up

Replace i² with -1

Simplify

2) i^4

$= i^2 \times i^2$

$= -1 \times -1$

$= 1$

Split up

Replace the i² terms with -1

Simplify

COMPLEX NUMBERS

You can multiply complex numbers and simplify powers of I

Complex numbers can be multiplied using the same techniques as used in algebra.

You can also use the following rule to simplify powers of i:

$i = \sqrt{-1}$

$i^2 = -1$

Simplify:

3) $(2i)^5$

$= 2^5 \times i^5$

Write both as a power of 5

$= 2^5 \times i^2 \times i^2 \times i$

Split up the i terms

$= 32 \times -1 \times -1 \times i$

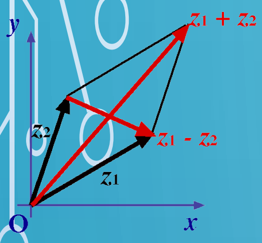
Work out 2^5 and replace the i^2 terms

$= 32i$

Simplify

1B

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

You can write down the complex conjugate of a complex number, and it helps you divide one complex number by another

If a complex number is given by:

a + bi

Then the complex conjugate is:

a – bi

(You just reverse the sign of the imaginary part!)

Together, these are known as a **complex conjugate pair**

The complex conjugate of z is written as z*

Write down the complex conjugate of:

a) $2 + 3i$

$= 2 - 3i$

Reverse the sign of the imaginary term

b) $5 - 2i$

$= 5 + 2i$

Reverse the sign of the imaginary term

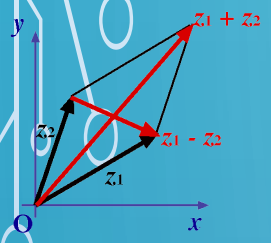
c) $1 - i\sqrt{5}$

$= 1 + i\sqrt{5}$

Reverse the sign of the imaginary term

1C

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

Find z + z*, and zz*, given that:

z = 2 - 7i

→ z* = 2 + 7i

$$z + z^*$$
$$= (2 - 7i) + (2 + 7i)$$
$$= 4$$

Replace z and z*

Group terms

$$zz^*$$
$$= (2 - 7i)(2 + 7i)$$
$$= 4 + 14i - 14i - 49i^2$$
$$= 4 - 49(-1)$$
$$= 53$$

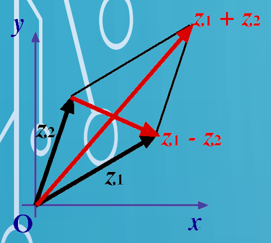
Replace z and z*

Multiply out

The i terms cancel out, replace i^2 with -1

Simplify

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

Simplify $(10 + 5i) \div (1 + 2i)$

With divisions you will need to write it as a fraction, then multiply both the numerator and denominator by the complex conjugate of the denominator

(This is effectively the same as rationalising when surds are involved!)

$$\frac{10+5i}{1+2i} \times \frac{1-2i}{1-2i}$$

Multiply by the complex conjugate of the denominator

$$= \frac{(10+5i)(1-2i)}{(1+2i)(1-2i)}$$

Expand both brackets

$$= \frac{10+5i-20i-10i^2}{1+2i-2i-4i^2}$$

Group i terms, replace the i^2 terms with -1 (use brackets to avoid mistakes)

$$= \frac{10-15i-10(-1)}{1-4(-1)}$$

Simplify terms

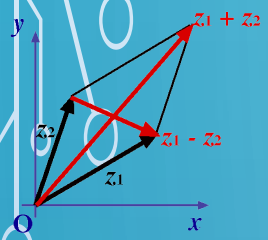
$$= \frac{20-15i}{5}$$

Divide by 5

$$= 4 - 3i$$

1C

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

Simplify:

$(5+4i) \div (2-3i)$

With divisions you will need to write it as a fraction, then multiply both the numerator and denominator by the complex conjugate of the denominator

(This is effectively the same as rationalising when surds are involved!)

$$\frac{5+4i}{2-3i} \times \frac{2+3i}{2+3i}$$

Multiply by the complex conjugate of the denominator

$$= \frac{(5+4i)(2+3i)}{(2-3i)(2+3i)}$$

Expand both brackets

$$= \frac{10+8i+15i+12i^2}{4+6i-6i-9i^2}$$

Group i terms, replace the i^2 terms with -1 (use brackets to avoid mistakes)

$$= \frac{10+23i+12(-1)}{4-9(-1)}$$

Simplify terms

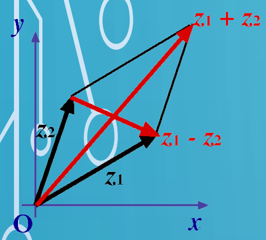
$$= \frac{-2+23i}{13}$$

Split into two parts (this is useful for later topics!)

$$= -\frac{2}{13} + \frac{23}{13}i$$

1C

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

If the roots a and b of a quadratic equation are complex, a and b will always be a complex conjugate pair

→You can find what a quadratic equation was by using its roots

→Let us start by considering a quadratic equation with real solutions…

$x^2 + 7x + 10 = 0$

Factorise

$(x+5)(x+2) = 0$

Solve

$Roots\ are\ -5\ and\ -2$

Add the roots together

$(-5) + (-2)$

$= -7$

Adding the roots gives the negative of the 'b' term

Multiply the roots

$(-5) \times (-2)$

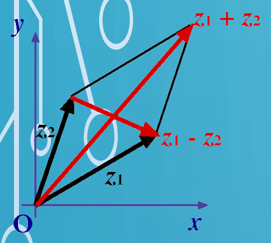
$= 10$

Multiplying the roots gives the 'c' term

This will work every time!
→ If you have the roots of a quadratic equation:
→ Add them and reverse the sign to find the 'b' term
→ Multiply them to find the 'c' term

1C

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

If the roots a and b of a quadratic equation are complex, a and b will always be a complex conjugate pair

→You can find what a quadratic equation was by using its roots

→Let us start by considering a quadratic equation with real solutions…

$x^2 + 2x - 24 = 0$

Factorise

$(x + 6)(x - 4) = 0$

Solve

$Roots\ are\ -6\ and\ 4$

<u>Add the roots together</u>

$(-6) + (4)$

$= -2$

Adding the roots gives the <u>negative</u> of the 'b' term

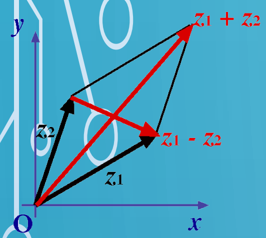
<u>Multiply the roots</u>

$(-6) \times (4)$

$= -24$

Multiplying the roots gives the 'c' term

COMPLEX NUMBERS



You can find the complex conjugate of a complex number

Find the quadratic equation that has roots 3 + 5i and 3 – 5i

Add the roots together

$(3+5i)+(3-5i)$

$=6$

Group terms

$So\ the'b'term\ is\ -6$

Multiply the roots

$(3+5i)(3-5i)$

Multiply out brackets

$=9+15i-15i-25i^2$

Group the 'i' terms, replace i^2 with -1

$=9-25(-1)$

Simplify

$=34$

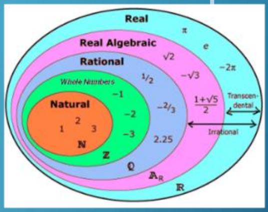
$So\ the'c'term\ is\ 34$

Now you have the b and c coefficients, you can write the equation!

$The\ equation\ is\ therefore:$

$x^2-6x+34=0$

COMPLEX NUMBERS



You can represent complex numbers on an Argand diagram

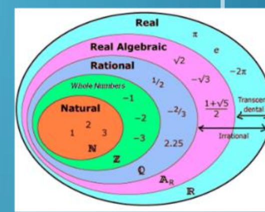
A grid where values for x and y can be plotted is known as a Cartesian set of axes (after Rene Descartes)

An Argand diagram is very similar, but the x-axis represents real numbers and the y-axis represents imaginary numbers.

Complex numbers can be plotted on an Argand diagram, by considering the real and imaginary parts as coordinates

1D

COMPLEX NUMBERS



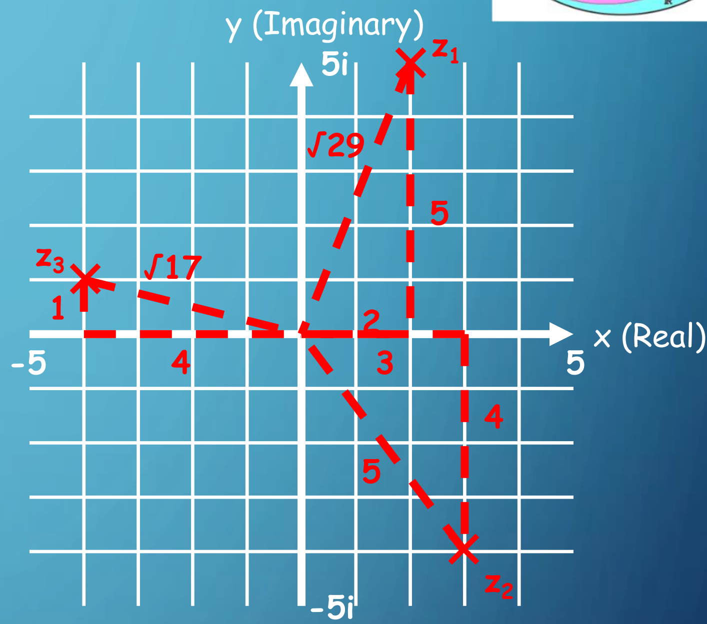
You can represent complex numbers on an Argand diagram

Represent the following complex numbers on an Argand diagram:

$$z_1 = 2 + 5i$$

$$z_2 = 3 - 4i$$

$$z_3 = -4 + i$$



Find the magnitude of |OA|, |OB| and |OC|, where O is the origin of the Argand diagram, and A, B and C are z_1, z_2 and z_3 respectively

$$|OA| = \sqrt{2^2 + 5^2}$$

$$|OA| = \sqrt{29}$$

$$|OB| = \sqrt{3^2 + 4^2}$$

$$|OB| = 5$$

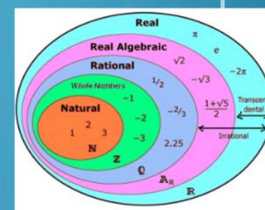
$$|OC| = \sqrt{4^2 + 1^2}$$

$$|OC| = \sqrt{17}$$

→ You can use Pythagoras' Theorem to find the magnitude of the distances

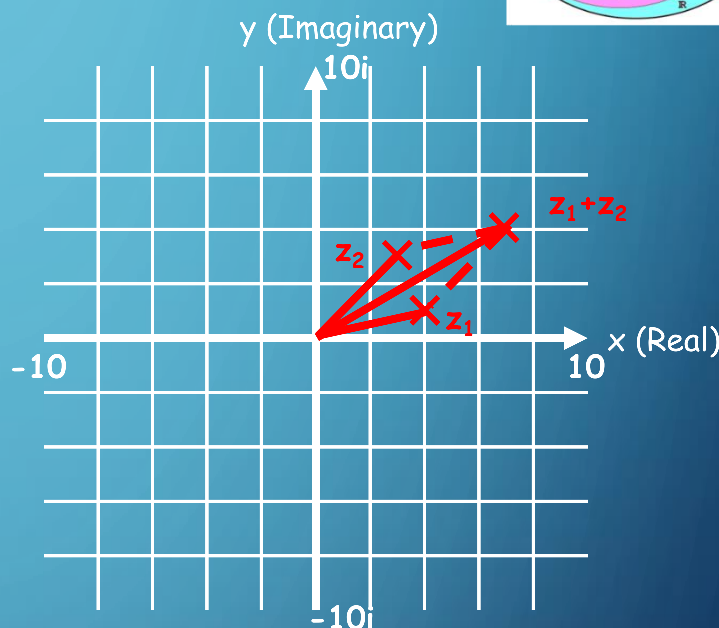
1D

COMPLEX NUMBERS



You can represent complex numbers on an Argand diagram

$z_1 = 4 + i$ $\qquad$ $z_2 = 3 + 3i$



Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram

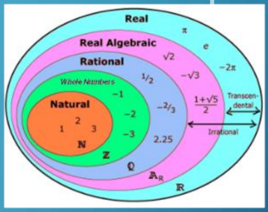
$$(4 + i) + (3 + 3i)$$

$$= 7 + 4i$$

Notice that vector $z_1 + z_2$ is effectively the diagonal of a parallelogram

1D

COMPLEX NUMBERS



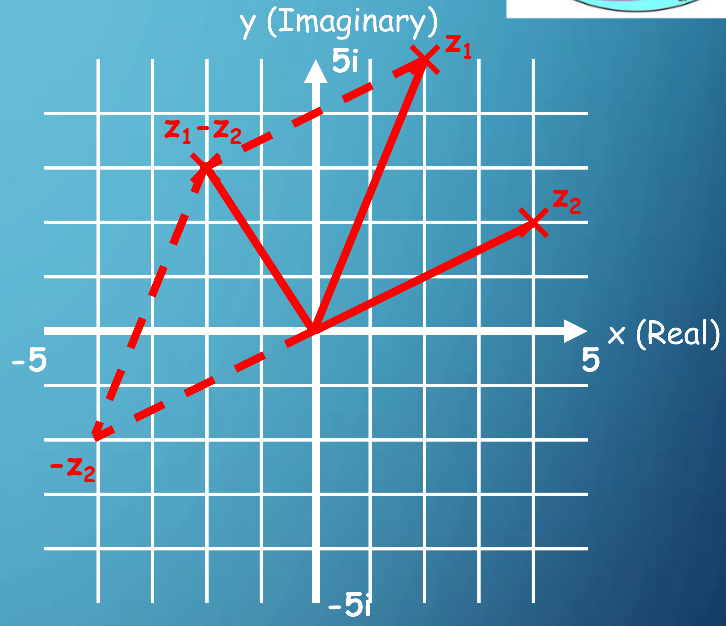
You can represent complex numbers on an Argand diagram

$z_1 = 2 + 5i \qquad z_2 = 4 + 2i$

Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram

$(2 + 5i) - (4 + 2i)$

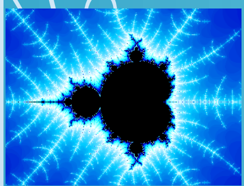
$= -2 + 3i$



Vector $z_1 - z_2$ is still the diagram of a parallelogram

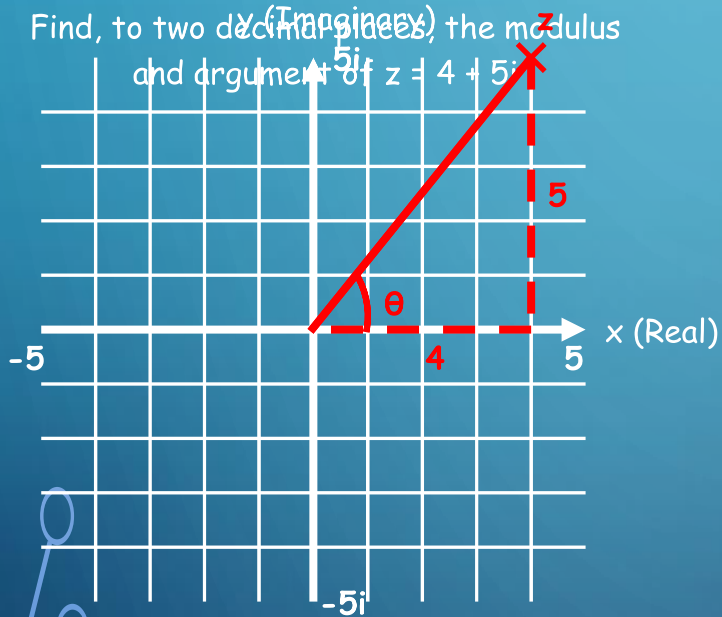
→ One side is z_1 and the other side is $-z_2$ (shown on the diagram)

COMPLEX NUMBERS



You can find the value of r, the modulus of a complex number z, and the value of θ, which is the argument of z

Find, to two decimal places, the modulus and argument of z = 4 + 5i



Use Pythagoras' Theorem to find r

$$r = \sqrt{4^2 + 5^2}$$

Calculate

$$r = \sqrt{41}$$

Work out as a decimal (if needed)

$$r = 6.40\ (2dp)$$

Use Trigonometry to find arg z

$$Tan\theta = \frac{O}{A}$$

Sub in values

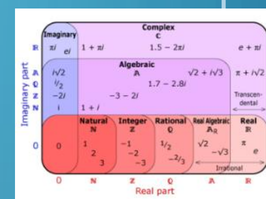
$$Tan\theta = \frac{5}{4}$$

Calculate in radians

$$\theta = 0.90\ radians\ (2dp)$$

1E

COMPLEX NUMBERS



You can find the modulus-argument form of the complex number z

You have seen up to this point that a complex number z will usually be written in the form:

$$z = x + iy$$

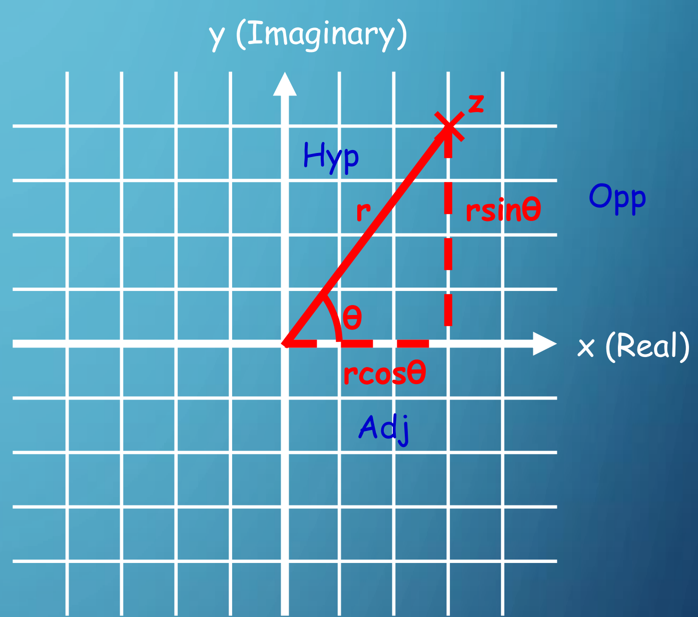
The modulus-argument form is an alternative way of writing a complex number, and it includes the modulus of the number as well as its argument.

The modulus-argument form looks like this:

$$z = r(cos\theta + isin\theta)$$

r is the modulus of the number

θ is the argument of the number



By GCSE Trigonometry:

O
S H → $Opp = Hyp \times Sin\theta \quad = rsin\theta$

A
C H → $Adj = Hyp \times Cos\theta \quad = rcos\theta$

$z = rcos\theta + irsin\theta$

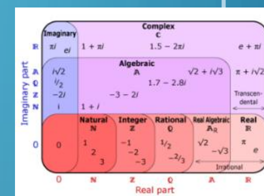
$z = r(cos\theta + isin\theta)$

Factorise

1F

COMPLEX NUMBERS

$z = r(cos\theta + isin\theta)$



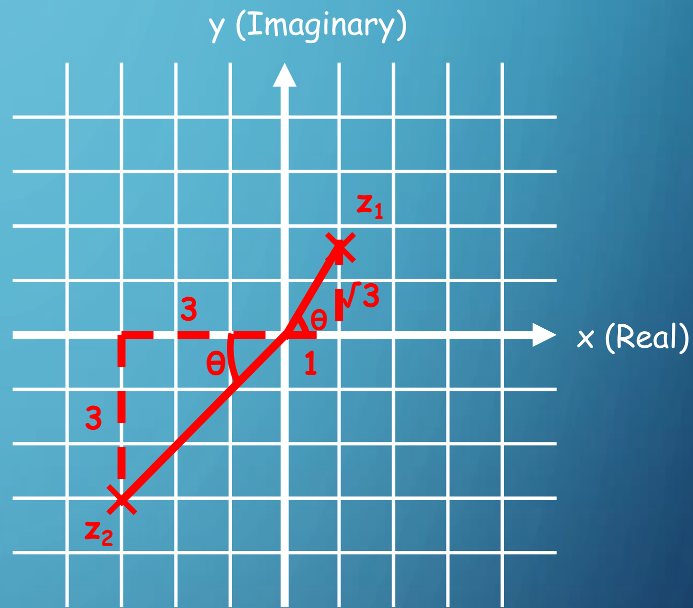
You can find the modulus-argument form of the complex number z

Express the numbers following numbers in the modulus argument form:

$z_1 = 1 + i\sqrt{3}$

$z_2 = -3 - 3i$

$z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$



<u>Modulus for z_1</u>

$\sqrt{1^2 + \sqrt{3}^2}$

$= 2$

<u>Argument for z_1</u>

$Tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \frac{\pi}{3}$

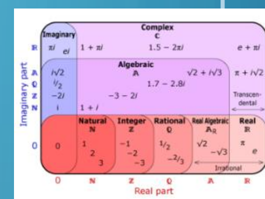
$z_1 = r(cos\theta + isin\theta)$

$z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$

1F

COMPLEX NUMBERS

$z = r(\cos\theta + i\sin\theta)$



You can find the modulus-argument form of the complex number z

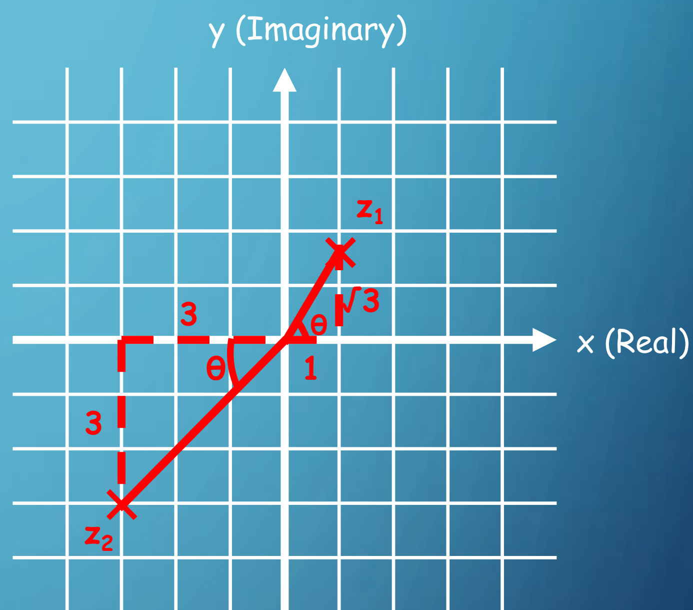
Express the numbers following numbers in the modulus argument form:

$z_1 = 1 + i\sqrt{3}$

$z_2 = -3 - 3i$

$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z_2 = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$



Modulus for z_2

$\sqrt{3^2 + 3^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

Argument for z_2

$Tan^{-1}\left(\frac{3}{3}\right)$

Remember the angle you actually want!

$= \frac{\pi}{4} \rightarrow = -\frac{3\pi}{4}$

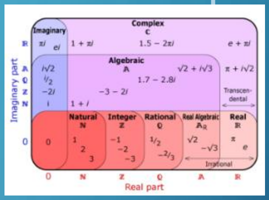
$z_2 = r(\cos\theta + i\sin\theta)$

$z_2 = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

1F

COMPLEX NUMBERS

$z = r(cos\theta + isin\theta)$



You can find the modulus-argument form of the complex number z

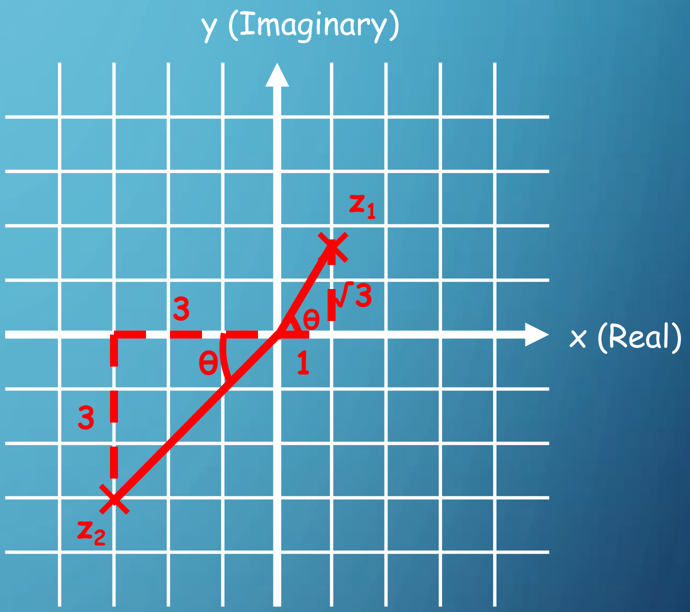
Express the numbers following numbers in the modulus argument form:

$$z_1 = 1 + i\sqrt{3}$$

$$z_2 = -3 - 3i$$

$$z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$$

$$z_2 = 3\sqrt{2}\left(cos\left(-\frac{3\pi}{4}\right) + isin\left(-\frac{3\pi}{4}\right)\right)$$



$$|z_1 z_2| \quad = |z_1||z_2|$$

Write down the value of $|z_1 z_2|$

$$= 2 \times 3\sqrt{2}$$

$$= 6\sqrt{2}$$

1F

$z = r(\cos\theta + i\sin\theta)$

You can find the modulus-argument form of the complex number z

A complex number is represented in the modulus-argument form as:

$$z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Write the number in the form:

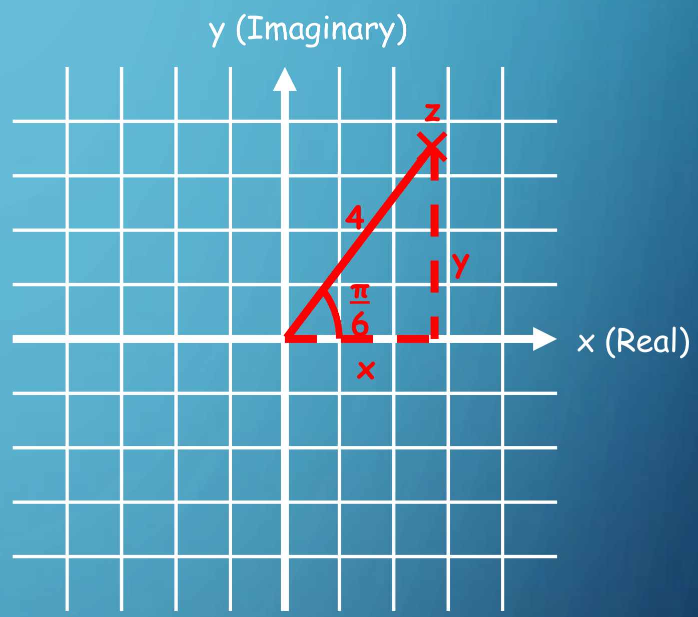
$$z = x + iy$$

Start by sketching the number on an Argand diagram

→ The modulus is 4

→ The angle is positive and less than $\pi/2$, so the point is somewhere in the top right section

→ Work out x and y using Trigonometry…



$x = r\cos\theta$

$x = 4\cos\frac{\pi}{6}$

$x = 2\sqrt{3}$

$y = r\sin\theta$

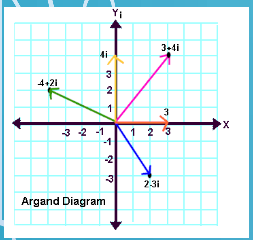
$y = 4\sin\frac{\pi}{6}$

$y = 2$

$So: \ z = 2\sqrt{3} + 2i$

Your sketch will help you decide whether answers are negative or positive!
It will also help you confirm what angle you should use…

COMPLEX NUMBERS



You can solve problems involving complex numbers

Problems can be solved by equating the real and imaginary parts of a complex equation

→ This technique can also be used to square root a number

$$3 + 5i = (a + ib)(1 + i)$$

Given that:

Find the real values of a and b

$$3 + 5i = (a + ib)(1 + i)$$

Multiply out the bracket

$$3 + 5i = a + ai + bi + i^2 b$$

Replace i^2

$$3 + 5i = a + ai + bi + (-1)b$$

Remove the bracket

$$3 + 5i = a + ai + bi + -b$$

Move the real and imaginary terms together

$$3 + 5i = a - b + ai + bi$$

Factorise the imaginary terms

$$\boxed{3} + \boxed{5}i = \boxed{a - b} + i\boxed{(a + b)}$$

As the equations balance, the real and imaginary parts will be the same on each side

→ Compare them and form equations

1) $a - b = 3$

2) $a + b = 5$

Add the equations together

$$2a = 8$$

Solve for a

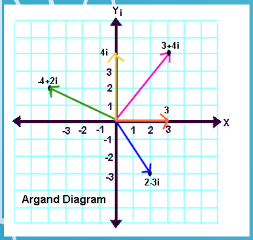
$$a = 4$$

Use a to find b

$$b = 1$$

1G

COMPLEX NUMBERS



You can solve problems involving complex numbers

Problems can be solved by equating the real and imaginary parts of a complex equation

→ This technique can also be used to square root a number

Find the square roots of 3 + 4i

→ Let the square root of 3 + 4i be given by a + ib

$$\sqrt{3+4i} = a + ib$$

Square both sides

$$3+4i = (a+ib)^2$$

Write as a double bracket

$$3+4i = (a+ib)(a+ib)$$

Multiply out the bracket

$$3+4i = a^2 + abi + abi + i^2b^2$$

Move real terms and imaginary terms together

$$\boxed{3} + \boxed{4i} = \boxed{a^2 - b^2} + \boxed{2abi}$$

As the equations balance, the real and imaginary parts will be the same on each side
→ Compare them and form equations

1) $a^2 - b^2 = 3$

2) $2ab = 4$

Divide by 2

$$ab = 2$$

Divide by a

$$b = \frac{2}{a}$$

1G

COMPLEX ANALYSIS

Complex Number: A complex number is defined by $z = x + iy$, the symbol 'z' is called a complex variable where x, y are real and $i = \sqrt{-1}$. x is the real part and y is the imaginary part of z.

Modules and argument of z:

If $z = x + iy$ then the modules of z is, $|z| = \sqrt{x^2 + y^2}$ and argument of z is, $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

❒ **Find the modules and argument of** $z = 1 - i$

Modules of z, $|z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

Now, Argument of z, $\arg(z) = \tan^{-1}\left(-\frac{1}{1}\right) = \tan(-1) = -45^\circ = 360^\circ - 45^\circ = 315^\circ$

❒ **Find the modules and argument of** $z = \frac{1+i}{1-i}$

Given, $z = \frac{1+i}{1-i} = \frac{(1+i)}{(1-i)}\frac{(1+i)}{(1+i)} = \frac{(1+i)^2}{1^2-i^2} = \frac{1+2i-1}{1+1} = \frac{2i}{2} = i$

Modules of z, $|z| = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

Argument of z , $\arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2} = 90°$

❒ **If $z = \frac{2+i}{2-i}$, find modules and argument of z**

Given that, $z = \frac{2+i}{2-i} = \frac{(2+i)}{(2-i)}\frac{(2+i)}{(2+i)} = \frac{(2+i)^2}{2^2-i^2} = \frac{4+2.2.i+i^2}{4+1} = \frac{3+4i}{5} = \frac{3}{5} + \frac{4}{5}i$

Modules of z , $|z| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9+16}{25}} = \sqrt{1} = 1$

Argument, $\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{\frac{4}{5}}{\frac{3}{5}}\right) = tan^{-1}\left(\frac{4}{3}\right) = 53.13°$

Argument of z , $\arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2} = 90°$

Exercise: Find the modules and argument of the following:

$$(i)\quad z = 2+i \quad (ii)\quad z = 1-3i \quad (iii)\quad z = (1+i)^2$$

$$(iv)\quad z = \frac{2-i}{3+i} \quad (v)\quad z = \frac{3+i}{1+i}$$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

Making a direct substitution, we have

$$z = x + yi$$
$$z = (r\cos\theta) + i(r\sin\theta)$$
$$z = r(\cos\theta + i\sin\theta)$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Euler Identity $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{ix} = 1 + ix + \frac{-1}{2!}x^2 + \frac{-i}{3!}x^3 + \cdots$$

$$\cos x = 1 + \frac{-1}{2!}x^2 + \cdots$$

$$i\sin x = ix + \frac{-i}{3!}x^3 + \cdots$$

Using identity, write $z = |z|e^{iarg(z)}$ (Polar form)

Find the polar form of $-4+4i$.

Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)$.

Week 2

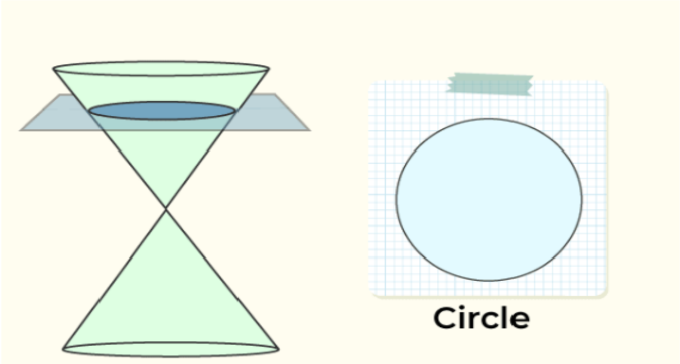
Topics : Geometrical representations

Page no (40-44)

1. Circle

The circle is a conic section in which it is the locus of the point that is always equidistant from the center of one point. If the plane cuts the conic section at right angles, i.e. β = 90° then we get a **circle**. The image for the same is added below,

- ***The general equation of the circle is, $(x - h)^2 + (y-k)^2 = r^2$.***
- ***Coordinates of Focus**: The circle's focus is its center, (h,k)(h, k)(h,k).*
- ***Directrix**: Not applicable, as circles do not have directrices.*



2. Ellipse

If the plane cuts the conic section at an angle less than 90°, i.e. α < β < 90° then we get an **ellipse**. We define the parabola as the locus of all the points that the sum of distance from two fixed points (focus) is always contact. The image for the same is added below,

- ***Standard Equation**:*
  - *Horizontal Major Axis:* $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
  - *Vertical Major Axis:* $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

Examples: Describe geometrically the region of the following functions,

(i) z-3z+3=2 (ii) z+3+z-3=10

Solution: (i) Given that, $\left|\frac{z-3}{z+3}\right| = 2$

$$\Rightarrow \frac{|z-3|}{|z+3|} = 2 \quad \therefore \ |z-3| = 2|z+3|$$

If, $z = x + iy$, then $|x + iy - 3| = 2|x + iy + 3|$

$$\Rightarrow \left|(x-3)+iy\right| = 2\left|(x+3)+iy\right|$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$$

Squaring both sides, we get,

$$(x-3)^2+y^2 = 4\left\{(x+3)^2+y^2\right\}$$

$$\Rightarrow x^2-6x+9+y^2 = 4\left(x^2+6x+9+y^2\right)$$

$$\Rightarrow x^2-6x+9+y^2 = 4x^2+24x+36+4y^2$$

$$\Rightarrow 3x^2+30x+3y^2+27=0$$

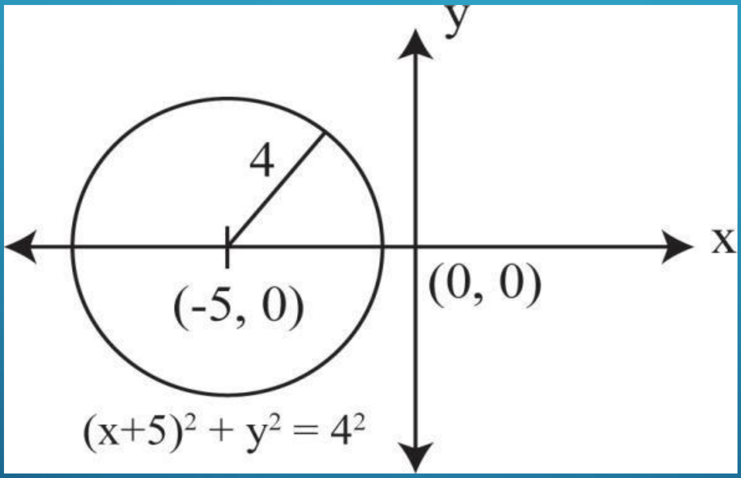
$$\Rightarrow x^2+10x+y^2+9=0$$

$$\Rightarrow x^2+2\cdot x\cdot 5+5^2+y^2 = 25-9$$

$$\Rightarrow (x+5)^2+y^2=16$$

$$\therefore\ (x+5)^2+y^2=4^2$$

which represents a circle of radius 4 and centre of (–5, 0)



(ii) Given that,

$$|z+3|+|z-3|=10$$

$$\Rightarrow |x+iy+3|+|x+iy-3|=10 \quad [\because\ z=x+iy]$$

$$\Rightarrow |(x+3)+iy|+|(x-3)+iy|=10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2}+\sqrt{(x-3)^2+y^2}=10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2}=10-\sqrt{(x-3)^2+y^2}$$

Squaring both sides we get,

$$(x+3)^2+y^2=100-20\sqrt{(x-3)^2+y^2}+(x-3)^2+y^2$$

$$\Rightarrow x^2+6x+9+y^2=100-20\sqrt{(x-3)^2+y^2}+x^2-6x+9+y^2$$

$$\Rightarrow 20\sqrt{(x-3)^2+y^2}=100-12x$$

$$\Rightarrow 5\sqrt{(x-3)^2+y^2}=25-3x$$

Again squaring both sides, we get,

$$\Rightarrow 25\left\{(x-3)^2+y^2\right\}=(25-3x)^2$$

$$\Rightarrow 25\left(x^2-6x+9+y^2\right)=625-150x+9x^2$$

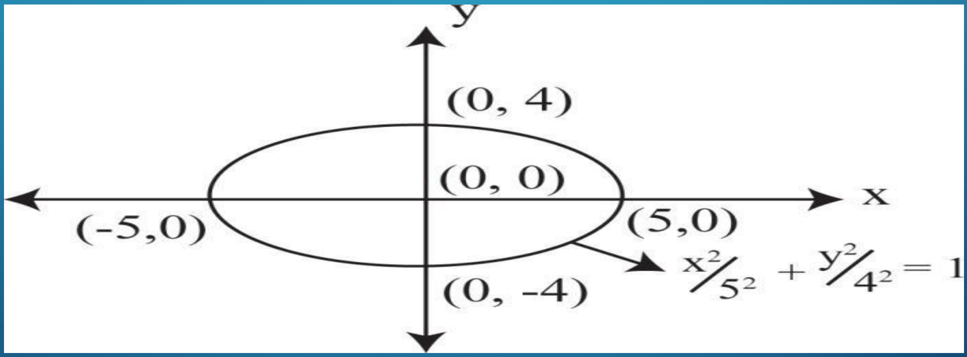
$$\Rightarrow 25x^2-150x+225+25y^2=625-150x+9x^2$$

$$\Rightarrow 16x^2+25y^2=400$$

$$\Rightarrow \frac{16}{400}x^2+\frac{25}{400}y^2=1$$

$$\therefore \frac{x^2}{5^2}+\frac{y^2}{4^2}=1$$

which represents an ellipse passing through the points $(\pm 5, 0)$ and $(0, \pm 4)$



Exercise: Describe geometrically the region of the following functions

(i) $|z-i|=2$ (ii) $|z+2i|+|z-2i|=6$ (iii) $|z-2|=4|z+2|$

(iv) $\left|\frac{z-1}{z+1}\right|=6$ (v) $|z+2|+|z-2|=3$

Week 3
Topics: Entire function, analytic function, Cauchy's Integral Formula
Page no (44-52)

Definition of Analytic Function:

A function f(z) is said to be analytic at a point z if z is an interior point of some region where f(z) is analytic. Hence the concept of analytic function at a point implies that the function is analytic in some circle with center at this point.

Definition of Entire Function

1. Entire Functions are related to the field of complex analysis, which is also called Integral Function.
2. An entire function is a complex-valued function that is a complex differential in a neighborhood of each point in a domain in a complex coordinate space, also known as holomorphic on the whole complex plane.
3. Every entire function can be represented as a power series.

Examples of Entire Function:
Polynomials and Exponential Functions are the entire functions as they are holomorphic on the whole complex plane.

Cauchy's Integral Formula:

Let $f(z)$ is analytic inside and on the boundary of a simply connected closed curve C and a is any point inside C. The $f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\mathrm{dz} \quad \Rightarrow \quad \oint_C \frac{f(z)}{z-a}\mathrm{dz}=2\pi i\times f(a)$

Where C is traversed in the positive sense.

Question: Evaluate $\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ Where C is the circle $|z|=3$

Solution: Let $\frac{1}{(z-1)(z-2)} = \frac{A}{z-2} + \frac{B}{z-1} \quad \Rightarrow 1 = A(z-1) + B(z-2) \quad \cdots \quad (1)$

Putting $z=2$ and $z=1$ in (1) we get $A=1$ and $B=-1$

Therefore $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

Hence $\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)} dz - \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)} dz \quad \cdots \quad (2)$

Let $f(z) = \sin\pi z^2 + \cos\pi z^2$. Since $z=1$ and $z=2$ are inside C and $f(z)$ is analytic inside and on C, so use com apply Cauchy's Integral formulae. Hence by Cauchy's integral formula we have $\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)} dz = 2\pi i\left\{\sin\pi(1)^2 + \cos\pi(1)^2\right\} = 2\pi i(\sin\pi + \cos\pi) = -2\pi i$

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)} dz = 2\pi i\left\{\sin\pi(2)^2 + \cos\pi(2)^2\right\} = 2\pi i(\sin 4\pi + \cos 4\pi) = 2\pi i$$

Substituting those values in (2), we get

$$\therefore \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = 2\pi i + 2\pi i = 4\pi i$$

Example: Evaluate $\oint_c \frac{e^{-z}}{z-1} dz$, where C is the circle $|z| = 2$

Solution: Here $f(z) = e^{-z}$ is an analytic function. So $f(z)$ is analytic inside and on the circle $|z| = 2$. Also $z=1$. $\therefore |z| = 1 < 2$. The point $z=1$ lies inside the circle $|z| = 2$.

Hence by Cauchy integral formula we have $\oint_c \frac{e^{-z}}{z-1} dz = 2\pi i \times f(1) = 2\pi i \times e^{-1} = \frac{2\pi i}{e}$

Question: Show that $\oint_c \frac{e^{tz}}{z^2+1}dz = 2\pi i \sin t$, where C is the circle $|z|=3$ and $t>0$

Solution: Here $f(z)=e^{tz}$ is analytic inside and on the given circle $|z|=3$

Again $z^2+1=z^2-(-1)=z^2-i^2=(z+i)(z-i)$

Let $\frac{1}{(z+i)(z-i)}=\frac{A}{z-i}+\frac{B}{z+i} \quad \Rightarrow 1=A(z+i)+B(z-i) \quad \cdots (1)$

Putting $z=i$ and $z=-i$ in (1) we get $A=\frac{1}{2i}$ and $B=-\frac{1}{2i}$

Therefore $\frac{1}{z^2+1}=\frac{1}{2i}\left\{\frac{1}{z-i}-\frac{1}{z+i}\right\}$

Therefore $\oint_c \frac{e^{tz}}{z^2+1}dz=\frac{1}{2i}\oint_c \frac{e^{tz}}{z-i}dz-\frac{1}{2i}\oint_c \frac{e^{tz}}{(z+i)}dz \quad \cdots \quad (2)$

Hence by Cauchy's integral formula we get

$\oint_c \frac{e^{tz}}{z-i}dz=2\pi i\times f(i)=2\pi i e^{it}$ and $\oint_c \frac{e^{tz}}{z+i}dz=2\pi i\times f(-i)=2\pi i e^{-it}$

Substituting that value in (2), we get

$$\oint_c \frac{e^{tz}}{z^2+1}dz=\frac{1}{2i}\left(2\pi i\, e^{it}\right)-\frac{1}{2i}\left(2\pi i\, e^{-it}\right)=2\pi i\left(\frac{e^{it}-e^{-it}}{2i}\right)=2\pi i \,\mathrm{sint}$$

❐ **Evaluate the following**:

$(i)\ \oint_C \frac{e^{3z}}{z-\pi i} \quad C:|z|=4 \qquad (ii)\ \oint_C \frac{\sin 3z}{z+\frac{\pi}{2}} \quad C:|z|=5 \qquad (iii)\ \oint_C \frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)} \quad C:|z|=3$

Cauchy-Riemann conditions

Analytic functions: If $f(z)$ is differentiable at $z = z_0$ and within the neighborhood of $z=z_0$, $f(z)$ is said to be **analytic** at $z = z_0$. A function that is analytic in the whole complex plane is called an *entire function*.

Cauchy-Riemann conditions for differentiability

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann conditions

Conversely, if the Cauchy-Riemann conditions are satisfied, $f(z)$ is differentiable:

$$\frac{df}{dz} = \lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y)}{\Delta x + i\Delta y} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}, \qquad \text{and } = \frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right).$$

More about Cauchy-Riemann conditions:

1) It is a very strong restraint to functions of a complex variable.

2) $\frac{df}{dz} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial (iy)}+i\frac{\partial v}{\partial (iy)}.$

3) $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \Rightarrow \nabla u\cdot\nabla v = 0 \Rightarrow \nabla u \perp \nabla v \Rightarrow u = c_1 \perp v = c_2$

4) Equivalent to $\frac{\partial f}{\partial z^*} = 0$, so that $f(z, z^*)$ only depends on z:

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \frac{\partial f}{\partial x}\frac{1}{2}+\frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right) = 0 \Rightarrow \frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y} = 0 \Rightarrow \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right) = 0 \Rightarrow \cdots$$

e.g., $f = x - iy$ is everywhere continuous but not analytic.

Cauchy's theorem

Cauchy's integral theorem

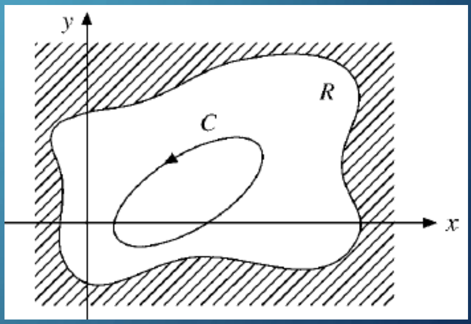
Contour integral:

$$\int_{z_1}^{z_2} f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

Cauchy's integral theorem: If $f(z)$ is analytic in a simply connected region R, [and $f'(z)$ is continuous throughout this region,] then for any closed path C in R, the contour integral of $f(z)$ around C is zero: $\oint_c f(z)dz = 0$

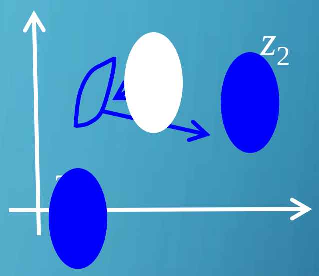
Proof using Stokes' theorem: $\oint_c \mathbf{V}\cdot d\boldsymbol{\lambda} = \iint_S \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}$

$$\oint_C (V_x dx + V_y dy) = \iint_S \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy$$



$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$
$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$
$$= 0$$

Cauchy-Goursat proof: The continuity of $f'(z)$ is not necessary.

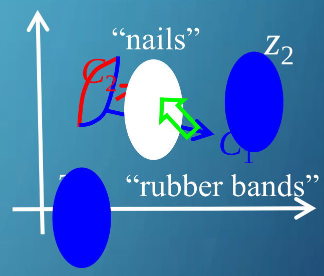


Corollary: An open contour integral for an analytic function is independent of the path, if there is no singular points between the paths.

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz$$

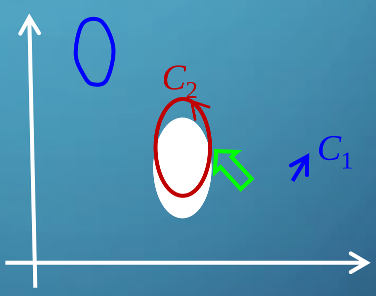
Contour deformation theorem:

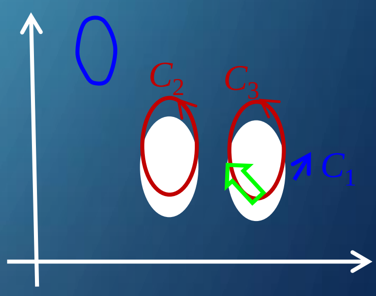
A contour of a complex integral can be arbitrarily deformed through an analytic region without changing the integral.



1) It applies to both open and closed contours.
2) One can even split closed contours.
Proof: Deform the contour bit by bit.

Examples:
1) Cauchy's integral theorem.
(Let the contour shrink to a point.)
2) Cauchy's integral formula.
(Let the contour shrink to a small circle.)





Cauchy's integral formula

Cauchy's integral formula:

If $f\,(z)$ is analytic within and on a closed contour C, then for any point z_0 within C,

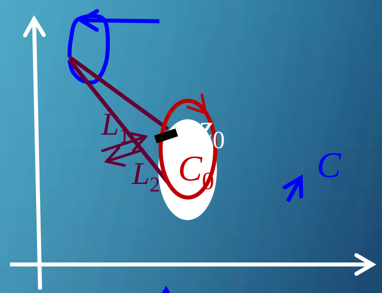
$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz$$

Proof:

$$\oint_C \frac{f(z)}{z - z_0}dz + \oint_{L_1} \frac{f(z)}{z - z_0}dz + \oint_{C_0} \frac{f(z)}{z - z_0}dz + \oint_{L_2} \frac{f(z)}{z - z_0}dz = 0$$

$$\oint_C \frac{f(z)}{z - z_0}dz = -\oint_{C_0} \frac{f(z)}{z - z_0}dz = -\int_{2\pi}^{0} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta}\,d\theta \quad (\text{Let } r \to 0)$$

$$= 2\pi i f(z_0)$$



Can directly use the contour deformation theorem.

Derivatives of $f(z)$: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

Corollary: If a function is analytic, then its derivatives of all orders exist.
Corollary: If a function is analytic, then it can be expanded in Taylor series.

Cauchy's inequality: If $f(z) = \sum a_n z^n$ is analytic and bounded, $|f(z)|_{|z|=r} \le M$,

then $|a_n| r^n \le M$. (That is, a_n is bounded.)

Proof: $f^{(n)}(0) = n!\, a_n = \frac{n!}{2\pi i}\oint_{|z|=r}\frac{f(z)}{z^{n+1}}dz \Rightarrow |a_n| = \frac{1}{2\pi}\left|\oint_{|z|=r}\frac{f(z)}{z^{n+1}}dz\right| \le \frac{M}{r^n} \Rightarrow |a_n| r^n \le M$

Liouville's theorem: If a function is analytic and bounded in the entire complex plane, then this function is a constant.

Proof: $|a_n| \le \frac{M}{r^n}$, let $r \to \infty$, then $a_n = 0$ for $n > 0$. $f(z) = a_0$.

Fundamental theorem of algebra: $P(z) = \sum_{i=0}^{n} a_i z^i \quad (n > 0, a_n \ne 0)$ has n roots.

Suppose $P(z)$ has no roots, then $1/P(z)$ is analytic and bounded as $|z| \to \infty$. Then $P(z)$ is constant. That is nonsense. Therefore $P(z)$ has at least one root we can divide out.

Week 4
Topics: Harmonic and Conjugate harmonic Page no (53-58)

Harmonic Function: Harmonic functions occur regularly and play an essential role in maths and other domains like physics and engineering. In complex analysis, harmonic functions are called the solutions of the Laplace equation. Every harmonic function is the real part of a holomorphic function in an associated domain. In this article, you will learn the definition of harmonic function, along with some fundamental properties.

Before learning about harmonic functions, let's recall the definition of the Laplace equation. An equation

having the second-order partial derivatives of the form.

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. This is usually written as

$$\nabla^2 f = 0$$

or

$$\Delta f = 0$$

Question: Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is a harmonic function and hence finds its harmonic conjugate v if $f(z) = u + iv$ is analytic.

Solution: Given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\right)$$

$$= 3x^2 - 3y^2 + 6x = \phi_1(x, y), say \qquad (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\right)$$
$$= -6xy - 6y = \phi_2(x, y), say \quad \cdots\cdots \quad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 - 3y^2 + 6x\right) = 6x + 6 \quad \cdots\cdots \quad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-6xy - 6y\right) = -6x - 6 \quad \cdots\cdots \quad (4)$$

Adding (3) and (4) we get, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0 \quad \Rightarrow \nabla^2 u = 0$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z,\ y = 0$ in (1) and (2), we get $\phi_1(\mathrm{z}, 0) = 3z^2 + 6z$ and $\phi_2(\mathrm{z}, 0) = 0$

By Milne's method we have $f'(z) = \phi_1(\mathrm{z}, 0) - i\,\phi_2(\mathrm{z}, 0) = 3z^2 + 6z$

So $f(z) = \int\left(3z^2 + 6z\right)dz = z^3 + 3z^2 + c_1 + ic_2$, where $c_1 + ic_2$ is a complex constant.

Then

$$u + iv = (x + iy)^3 + 3(x + iy)^2 + c_1 + ic_2$$
$$u + iv = x^3 + 3x^2iy + 3x\,i^2y^2 + i^3y^3 + 3\left(x^2 + 2xiy + i^2y^2\right) + c_1 + ic_2$$
$$u + iv = x^3 + 3x^2iy - 3xy^2 - iy^3 + 3x^2 + 6ixy - 3y^2 + c_1 + ic_2$$
$$u + iv = x^3 - 3xy^2 + 3x^2 - 3y^2 + c_1 + i\left(3x^2y - y^3 + 6xy + c_2\right)$$

Equating imaginary part from both sides we get

$$v = 3x^2y - y^3 + 6xy + c_2$$

<u>Question:</u> Show that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is a harmonic function and hence finds its harmonic conjugate v if $f(z) = u + iv$ is analytic.

Solution: Given that $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y + 2x^2 - y^3 - 2y^2\right)$$
$$= 6xy + 4x = \phi_1(x, y), say \qquad (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y + 2x^2 - y^3 - 2y^2\right)$$
$$= 3x^2 - 3y^2 - 4y = \phi_2(x, y), say \qquad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(6xy + 4x) = 6y + 4 \qquad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(3x^2 - 3y^2 - 4y\right) = -6y - 4 \qquad (4)$$

Adding (3) and (4) we get, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6x - 4 = 0 \quad \Rightarrow \nabla^2 u = 0$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z,\ y = 0$ in (1) and (2), we get $\phi_1(\mathrm{z}, 0) = 4z$ and $\phi_2(\mathrm{z}, 0) = 3z^2$

By Milne's method we have $f'(z) = \phi_1(\mathrm{z}, 0) - i\,\phi_2(\mathrm{z}, 0) = 4z - i3z^2$

So $f(z) = \int\left(4z - i3z^2\right)dz = 2z^2 - iz^3 + c_1 + ic_2$, where $c_1 + ic_2$ is a complex constant.

Then

$$u + iv = 2(x + iy)^2 - i(x + iy)^3 + c_1 + ic_2$$
$$u + iv = 2\left(x^2 + 2xiy + i^2y^2\right) - i\left(x^3 + 3x^2iy + 3x\,i^2y^2 + i^3y^3\right) + c_1 + ic_2$$
$$u + iv = 2x^2 + 4xiy - 2y^2 - ix^3 + 3x^2y + i3x\,y^2 - y^3 + c_1 + ic_2$$
$$u + iv = 3x^2y + 2x^2 - y^3 - 2y^2 + i\left(4xy - x^3 + 3xy^2 + c_2\right)$$

Equating imaginary part from both sides we get

$$v = 4xy - x^3 + 3xy^2 + c_2$$

Week 5

Topics: Finding conjugate harmonics Page no (57-59)

Question: Show that $u = x^3 + 6x^2y - 3xy^2 - 2y^3$ is a harmonic function and hence finds its harmonic conjugate v if $f(z) = u + iv$ is analytic.

Solution: Given that $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + 6x^2y - 3xy^2 - 2y^3\right)$$
$$= 3x^2 + 12xy - 3y^2 = \phi_1(x, y), say \quad \cdots \quad (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + 6x^2y - 3xy^2 - 2y^3\right)$$
$$= 6x^2 - 6xy - 6y^2 = \phi_2(x, y), say \quad \cdots \quad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 12xy - 3y^2\right) = 6x + 12y \quad \cdots \quad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(6x^2 - 6xy - 6y^2\right) = -6x - 12y \quad \cdots \quad (4)$$

Adding (3) and (4) we get, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y = 0 \quad \Rightarrow \nabla^2 u = 0$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z,\ y = 0$ in (1) and (2), we get $\phi_1(z, 0) = 3z^2$ and $\phi_2(z, 0) = 6z^2$

By Milne's method we have $f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0) = 3z^2 - i6z^2 = 3(1 - 2i)z^2$

So $f(z) = \int 3(1 - 2i)z^2\,dz = (1 - 2i)z^3 + c_1 + ic_2$, where $c_1 + ic_2$ is a complex constant.

Then

$$u+iv=(1-2i)(x+iy)^3+c_1+ic_2$$
$$u+iv=\left(x^3+3x^2iy+3x\,i^2y^2+i^3y^3\right)-2i\left(x^3+3x^2iy+3x\,i^2y^2+i^3y^3\right)+c_1+ic_2$$
$$u+iv=x^3+3x^2iy-3xy^2-iy^3-2ix^3-6x^2i^2y-6x\,i^3y^2-2i^4y^3+c_1+ic_2$$
$$u+iv=x^3+3x^2iy-3xy^2-iy^3-2ix^3+6x^2y+6i\,xy^2-2y^3+c_1+ic_2$$
$$u+iv=x^3+6x^2y-3xy^2-2y^3+c_1+i\left(3x^2y-2x^3-y^3+6xy^2+c_2\right)$$

Equating imaginary part from both sides we get $v=3x^2y-2x^3-y^3+6xy^2+c_2$

<u>Question:</u> Show that $u=e^x\left(x\cos y-y\sin y\right)$ is a harmonic function and hence finds its harmonic conjugate v if $f(z)=u+iv$ is analytic.

<u>Solution:</u> Given that $u=e^x\left(x\cos y-y\sin y\right)$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left[e^x\left(x\cos y-y\sin y\right)\right]=e^x\left(x\cos y-y\sin y\right)+e^x\cos y=\phi_1(x,y),say \qquad (1)$$

$$\therefore \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[e^x\left(x\cos y-y\sin y\right)\right]=e^x\left(-x\sin y-\sin y-y\cos y\right)=\phi_2(x,y),say \qquad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2}=e^x\left(x\cos y-y\sin y\right)+e^x\cos y+e^x\cos y$$
$$=e^x\left(x\cos y-y\sin y+2\cos y\right) \qquad (3)$$

$$\therefore \frac{\partial^2 u}{\partial y^2}=e^x\left(-x\cos y-\cos y-\cos y+y\sin y\right) \qquad (4)$$

Adding (3) and (4) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x\left(x\cos y-y\sin y+2\cos y\right)+e^x\left(-x\cos y-\cos y-\cos y+y\sin y\right)$$
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x\left(x\cos y-y\sin y+2\cos y-x\cos y-\cos y-\cos y+y\sin y\right)=e^x\times 0=0$$
$$\Rightarrow \nabla^2 u=0$$

Therefore, u satisfies Laplace equation, so u is a harmonic function.

Let v be the harmonic conjugate of u, so that $f(z)=u+iv$ is analytic.

Putting $x = z, y = 0$ in (1) and (2), we get $\phi_1(z,0) = e^z(z \cdot 1 - 0) + e^z \cdot 1 = ze^z + e^z$ & $\phi_2(z,0) = 0$

By Milne's method we have $f'(z) = \phi_1(z,0) - i\,\phi_2(z,0) = ze^z + e^z$

So $f(z) = \int (ze^z + e^z)dz = ze^z - \int e^z dz + e^z = ze^z + c_1 + ic_2$, $c_1 + ic_2$ is a complex constant.
Then

$$u + iv = (x+iy)e^{x+iy} + c_1 + ic_2$$
$$u + iv = (x+iy)e^x \cdot e^{iy} + c_1 + ic_2$$
$$u + iv = (x+iy)e^x(\cos y + i\sin y) + c_1 + ic_2$$
$$u + iv = e^x(x\cos y + ix\sin y + iy\cos y - y\sin y) + c_1 + ic_2$$
$$u + iv = e^x(x\cos y - y\sin y) + i\,e^x(x\sin y + y\cos y)$$

Equating imaginary part from both sides we get $v = e^x(x\sin y + y\cos y) + c_2$

Week 6:
Topics: Cauchy's Residue Page no (58-60)

Question: Show that $\oint_c \frac{e^{tz}}{(z^2+1)^2}dz$, where C is the circle $|z| = 3$ and $t > 0$

Solution: Here the circle $|z| = 3$. The poles of $\frac{e^{tz}}{(z^2+1)^2}$ are obtain by solving the equation
$(z^2+1)^2 = 0 \quad \Rightarrow z^2 + 1 = 0 \quad \Rightarrow z^2 - (-1) = 0 \quad \Rightarrow z^2 - i^2 = 0 \quad \Rightarrow (z+i)(z-i) = 0 \quad \therefore z = -i, i$

Both poles are double poles and lie inside the circle C, since $|i| = |-i| = 1 < 3$ and $|i| = 1 < 3$

Residue at $z = i$ is

$$\lim_{z\to i}\frac{1}{1!}\frac{d}{dz}\left[(z-i)^2\frac{e^{tz}}{(z-i)^2(z+i)^2}\right]=\lim_{z\to i}\frac{d}{dz}\left[\frac{e^{tz}}{(z+i)^2}\right]$$

$$=\lim_{z\to i}\frac{(z+i)^2\times te^{tz}-e^{tz}\times 2(z+i)}{(z+i)^4}$$

$$=\lim_{z\to i}\frac{t(z+i)e^{tz}-2e^{tz}}{(z+i)^3}=\frac{t(i+i)e^{it}-2e^{it}}{(i+i)^3}$$

$$=\frac{2it\,e^{it}-2e^{it}}{8i^3}=\frac{2(it-1)e^{it}}{-8i}$$

$$=\frac{(it-1)e^{it}}{-4i}=\frac{-(t+i)e^{it}}{4}$$

Similarly, residue at $z=-i$ is $\dfrac{-(t-i)e^{-it}}{4}$

Therefore, by Cauchy's residue theorem we have

$$\oint_c\frac{e^{tz}}{(z^2+1)^2}dz=2\pi i[sum\ \ of\ \ residue]=-2\pi i\left[\frac{(t+i)e^{it}+(t-i)e^{-it}}{4}\right]$$

$$=-2\pi i\left[\frac{t(e^{it}+e^{-it})+i(e^{it}-e^{-it})}{4}\right]$$

$$=-2\pi i\left[\frac{2t\cos t+i\times 2i\sin t}{4}\right]$$

$$=-\pi i\left[\frac{4t\cos t+4i^2\sin t}{4}\right]$$

$$=-\pi i(t\cos t-\sin t)=\pi i(\sin t-t\cos t)$$

Question: Show that $\displaystyle\oint_c\frac{e^{-iz}}{(z+3)(z-i)^2}dz$, where C is the circle $|z-1|=2$

Solution: Here the circle $|z|=3$. The poles of $\dfrac{e^{-iz}}{(z+3)(z-i)^2}$ are obtain by solving the equation

$$(z+3)(z-i)^2=0\quad\Rightarrow z+3=0\quad and\quad (z-i)^2=0\quad\Rightarrow z=-3\quad and\quad z=i,i$$

Since $|i|=1<2$ and $|-3|=3>2$. Thus the pole $z=i$ is double pole and lie inside the circle C.

Residue at $z=i$ is

$$\lim_{z\to i}\frac{1}{1!}\frac{d}{dz}\left[(z-i)^2\frac{e^{-iz}}{(z+3)(z-i)^2}\right]$$

$$=\lim_{z\to i}\frac{d}{dz}\left[\frac{e^{-iz}}{(z+3)}\right]$$

$$=\lim_{z\to i}\frac{(z+3)\times\left(-ie^{-iz}\right)-e^{-iz}\times 1}{(z+3)^2}$$

$$=\frac{(i+3)(-ie)-e}{(i+3)^2}=\frac{(1-3i-1)e}{-1+6i+9}$$

$$=\frac{-3ie}{8+6i}=\frac{-3ie(4-3i)}{2(4+3i)(4-3i)}$$

$$=\frac{-12ie+9i^2e}{2(16+9)}$$

$$=\frac{-12ie-9e}{50}$$

Therefore, by Cauchy's residue theorem we have

$$\oint_c\frac{e^{-iz}}{(z+3)(z-i)^2}dz=2\pi i\left[residue\ at\ z=i\right]$$

$$=2\pi i\times\left(\frac{-12ie-9e}{50}\right)$$

$$=\frac{-12i^2\pi e-9i\pi e}{50}$$

$$=\frac{(12-9i)\pi e}{25}$$

Week 7
Topics: Laurent series
Page no (61-70)

Laurent expansion:

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n, \quad a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

1) Singular points of the integrand.
   For $n<0$, the singular points are determined by $f(z)$. For $n\geq 0$, the singular points are determined by both $f(z)$ and $1/(z'-z_0)^{n+1}$.
2) If $f(z)$ is *analytic* inside C, then the Laurent series reduces to a Taylor series:

$$a_n=\begin{cases}\dfrac{f^{(n)}(z_0)}{n!}, & n\geq 0,\\ 0, & n<0.\end{cases}$$

3) Although a_n has a general contour integral form, In most times we need to use straight forward complex algebra to find a_n.

Laurent expansion: Examples

Example 1: Expand $f(z)=\dfrac{z^3}{(z-1)^2}$ about $z_0=1$.

$$\frac{z^3}{(z-1)^2}=\frac{[(z-1)+1]^3}{(z-1)^2}=\frac{(z-1)^3+3(z-1)^2+3(z-1)+1}{(z-1)^2}=\frac{1}{(z-1)^2}+\frac{3}{z-1}+3+(z-1)$$

Example 2: Expand $f(z)=\dfrac{1}{z^2+1}$ about $z_0=i$.

$$f(z)=\frac{1}{z^2+1}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{2i+z-i}\right)$$

$$=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{2i}\cdot\frac{1}{1+\frac{z-i}{2i}}\right)=\frac{1}{2i}\frac{1}{z-i}-\frac{1}{(2i)^2}\sum_{n=0}^{\infty}\left(-\frac{1}{2i}\right)^n(z-i)^n$$

$$=-\frac{i}{2}\frac{1}{z-i}+\frac{1}{4}+\frac{i}{8}(z-i)+\cdots$$

Branch points and branch cuts

Singularities

Poles: In a Laurent expansion $f(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m$, if $a_m = 0$ for $m < -n < 0$ and $a_{-n} \neq 0$,

then z_0 is said to be a *pole of order n*.
A pole of order 1 is called a *simple pole*.
A pole of infinite order (when expanded about z_0) is called an *essential singularity*.

The behavior of a function $f(z)$ at infinity is defined using the behavior of $f(1/t)$ at $t = 0$.

Examples:

1) $$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) = \frac{1}{2i}\left[-\frac{1}{z+i} - \frac{1}{2i-(z+i)}\right] = -\frac{1}{2i}\frac{1}{z+i} + \frac{1}{4}\frac{1}{1-(z+i)/2i}$$

$$= -\frac{1}{2i}\frac{1}{z+i} + \frac{1}{4}\left[1 + \frac{z+i}{2i} + \left(\frac{z+i}{2i}\right)^2 + \cdots\right]$$ has a single pole at $z = -i$.

2) $$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \sin\frac{1}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{t^{2n+1}}$$

sinz thus has an essential singularity at infinity.

3) z^2+1 has a pole of order 2 at infinity.

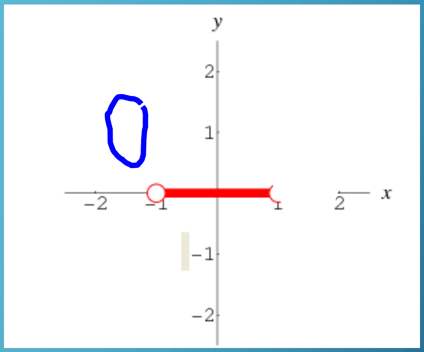
2. $f(z)=\sqrt{(z-1)(z+1)}$

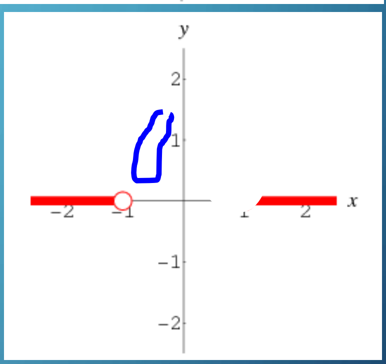
We can choose a branch cut from z = -1 to z = 1 (or any curve connecting these two points). The function will be single-valued, because both points will be circled.

Alternatively, we can choose a branch cut which connects each branch point to infinity. The function will be single-valued, because neither points will be circled.

It is notable that these two choices result in different functions. E.g., if $f(i)=\sqrt{2i}$, then

$f(-i)=-\sqrt{2}i$ for the first choice and $f(-i)=\sqrt{2}i$ for the second choice.

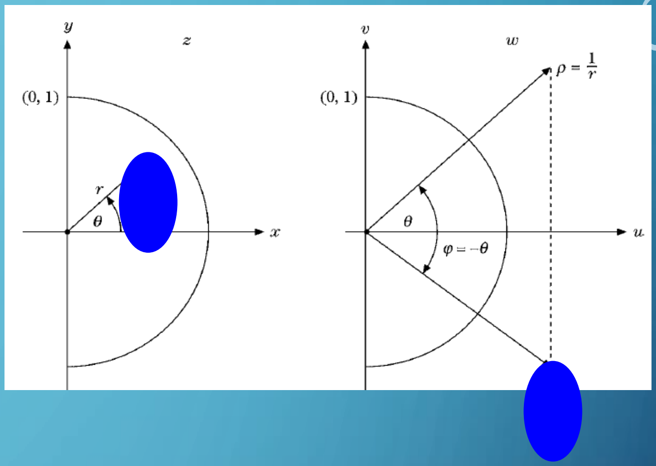




Inversion:

$$w=\frac{1}{z},\ \text{or}$$

$$\rho e^{i\varphi}=\frac{1}{re^{i\theta}}\Rightarrow\begin{cases}\rho=\dfrac{1}{r}\\ \varphi=-\theta\end{cases}$$



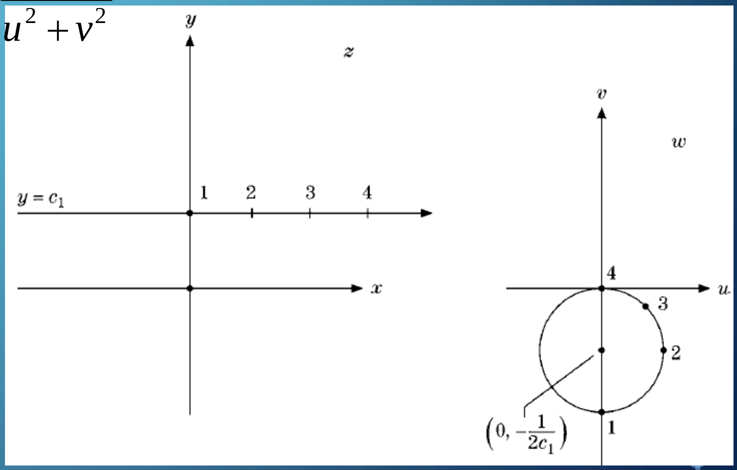
In Cartesian coordinates:

$$w=\frac{1}{z}\Rightarrow u+iv=\frac{1}{x+iy}\Rightarrow\begin{cases}u=\dfrac{x}{x^2+y^2}\\ v=-\dfrac{y}{x^2+y^2}\end{cases},\begin{cases}x=\dfrac{u}{u^2+v^2}\\ y=-\dfrac{v}{u^2+v^2}\end{cases}.$$

A straight line is mapped into a circle:

$$y=ax+b\Rightarrow-\frac{v}{u^2+v^2}=\frac{au}{u^2+v^2}+b$$

$$\Rightarrow b(u^2+v^2)+au+v=0.$$



Question: Expand the function $f(z)=\dfrac{1}{(z+1)(z+3)}$ in a **Laurent** series for the following region: $(i)\quad 1<|z|<3 \qquad (ii)\quad |z|<1$

Solution: Given $f(z)=\dfrac{1}{(z+1)(z+3)}=\dfrac{1}{(z+1)(-1+3)}+\dfrac{1}{(-3+1)(z+3)}=\dfrac{1}{2(z+1)}-\dfrac{1}{2(z+3)}$

$(i)\quad 1<|z|<3 \Rightarrow 1<|z| \quad and \quad |z|<3 \quad \Rightarrow \dfrac{1}{|z|}<1 \quad and \quad \dfrac{|z|}{3}<1$

We write $f(z)$ in a manner so that the binomial expansion is valid for

$1<|z|<3 \quad \Rightarrow \dfrac{1}{|z|}\, and\, \dfrac{|z|}{3}<1$

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

$$=\frac{1}{2z}\times\frac{1}{1+\frac{1}{z}}-\frac{1}{6}\times\frac{1}{1+\frac{z}{3}}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

$$=\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right)-\left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\cdots\right)$$

$(i)\quad |z|<1 \Rightarrow |z|<1 \quad and \quad |z|<3 \quad \Rightarrow |z|<1 \quad and \quad \dfrac{|z|}{3}<1$

We write $f(z)$ in a manner so that the binomial expansion is valid for $|z|<1 \quad and \quad \dfrac{|z|}{3}<1$

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

$$=\frac{1}{2}(1+z)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2}\left(1-z+z^2-z^3+\cdots\right)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

$$=\left(\frac{1}{2}-\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{1}{18}\right)z+\left(\frac{1}{2}-\frac{1}{54}\right)z^2+\left(-\frac{1}{2}+\frac{1}{162}\right)z^3+\cdots$$

$$=\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\frac{40}{81}z^3+\cdots$$

Question: Expand the function $f(z)=\dfrac{z^2+1}{(z+1)(z-2)}$ in a **Laurent** series for the following region: $(i)\quad 1<|z|<2 \qquad (ii)\quad 0<|z|<1$

Solution: Given $f(z)=\dfrac{z^2+1}{(z+1)(z-2)}$. Let $\dfrac{z^2+1}{(z+1)(z-2)}=1+\dfrac{a}{z+1}+\dfrac{b}{z-2}$

Therefore, $z^2+1=(z+1)(z-2)+a(z-2)+b(z+1)$

When $z=-1$, then $2=0+a(-3)+0 \quad \Rightarrow a=\dfrac{-2}{3}$. When $z=2$, then $5=0+0+3b \quad \Rightarrow b=\dfrac{5}{3}$

$$\therefore \quad f(z)=\frac{z^2+1}{(z+1)(z-2)}=1-\frac{2/3}{z+1}+\frac{5/3}{z-2} \quad \cdots \quad (1)$$

$(i)\quad 1<|z|<3 \Rightarrow 1<|z| \quad and \quad |z|<2 \quad \Rightarrow \dfrac{1}{|z|}<1 \quad and \quad \dfrac{|z|}{2}<1$

Therefore, from (1) we have

$$f(z)=1-\frac{2/3}{z\left(1+\frac{1}{z}\right)}+\frac{5/3}{-2\left(1-\frac{z}{2}\right)}$$

$$=1-\frac{2}{3z}\left(1+\frac{1}{z}\right)^{-1}-\frac{5}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$=1-\frac{2}{3z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)-\frac{5}{6}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$=\cdots+\frac{2}{3z^4}-\frac{2}{3z^3}+\frac{2}{3z^2}-\frac{2}{3z}+\frac{1}{6}-\frac{5z}{12}-\frac{5z^2}{24}-\frac{5z^3}{48}-\cdots$$

$(i)\quad 0<|z|<1 \Rightarrow |z|<1. \quad Also \quad |z|<1 \quad \Rightarrow |z|<2 \quad \Rightarrow \quad \frac{|z|}{2}<1$

Therefore, from (1) we have

$$f(z)=1-\frac{2/3}{(z+1)}+\frac{5/3}{-2\left(1-\frac{z}{2}\right)}$$

$$=1-\frac{2}{3}(1+z)^{-1}-\frac{5}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$=1-\frac{2}{3}\left(1-z+z^2-z^3+\cdots\right)-\frac{5}{6}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$=\left(1-\frac{1}{2}-\frac{5}{6}\right)+\left(\frac{2}{3}-\frac{5}{12}\right)z+\left(-\frac{2}{3}-\frac{5}{24}\right)z^2+\left(\frac{2}{3}-\frac{5}{48}\right)z^3+\cdots$$

$$=-\frac{1}{2}+\frac{1}{4}z-\frac{7}{8}z^2+\frac{9}{16}z^3+\cdots$$

Question: Solve the Partial Differential Equation $x\left(y^2-z^2\right)p+y^2(z-x)q=z\left(x^2-y^2\right)$

Solution: Given the partial differential equation is

$$x\left(y^2-z^2\right)p+y^2(z-x)q=z\left(x^2-y^2\right)$$

The Lagrange auxiliary equations of (1) are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)} \quad \cdots\cdots \quad (2)$$

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{xdx+ydy+zdz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-y^2z^2}=\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c}{2}$$

$$\therefore\ x^2+y^2+z^2=c \quad \cdots\cdots \quad (3)$$

Again (2), implies

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating, we get

$$\ln x+\ln y+\ln z=\ln c_1$$
$$\Rightarrow \ln xyz=\ln c_1$$
$$\therefore\ xyz=c_1 \quad \cdots\cdots \quad (4)$$

From (3) and (4), the required general solution is $x^2+y^2+z^2=\phi\left(xyz\right)$, where ϕ being an arbitrary function.

$$\left[\quad p=\frac{\partial}{\partial x},\quad q=\frac{\partial}{\partial y}\quad\right]$$

<u>Question:</u> Find the particular integrals of the following partial differential equation to represent surface passing through the given lines.

$$x\left(y^2+z\right)p-y\left(x^2+z\right)q=z\left(x^2-y^2\right);\quad x=t,\quad y=-t,\quad z=1$$

<u>Solution:</u> Given the partial differential equation is

$$x\left(y^2+z\right)p-y\left(x^2+z\right)q=z\left(x^2-y^2\right) \quad \cdots\cdots \quad (1)$$

And the line is $x=t,\quad y=-t,\quad z=1 \quad \cdots\cdots \quad (2)$

The Lagrange auxiliary equations of (1) are

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{z\left(x^2-y^2\right)} \quad \cdots\cdots \quad (3)$$

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{z\left(x^2-y^2\right)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}d=0$$

Integrating, we get

$$\ln x + \ln y + \ln z = \ln c_1$$
$$\Rightarrow \ln xyz = \ln c_1$$
$$\therefore\ xyz = c_1 \quad \cdots \quad (4)$$

Again (3), implies

$$\frac{dx}{x\left(y^2+z\right)} = \frac{dy}{-y\left(x^2+z\right)} = \frac{dz}{z\left(x^2-y^2\right)} = \frac{xdx+ydy-dz}{x^2y^2+x^2z-y^2x^2-y^2z-x^2z+y^2z} = \frac{xdx+ydy-dz}{0}$$
$$\Rightarrow xdx+ydy-dz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-z=\frac{c_2}{2}$$
$$\therefore\ x^2+y^2-z=c_2 \quad \cdots \quad (5)$$

By (2) and (4), $-t^2=c_1$ and (5) implies $2t^2-2=c_2 \quad \Rightarrow 2c_1+c_2+2=0 \quad \cdots \quad (6)$

Now, putting the values of c_1 and c_2 from (4) and (5) in (6), we obtain

$2xyz+x^2+y^2-2z+2=0$ which is the required solut

Week 8

Topics: 2-D

Geometry

Page (70-73)

Question: Transform to parallel axes through the point $(1,-1)$ of the following equation

$$17x^2+18xy-7y^2-16x-32y+18=0 \quad \cdots\cdots \quad (1)$$

Solution: The given equation is $17x^2+18xy-7y^2-16x-32y+18=0$

Putting $x=x'+1$ and $y=y'-1$ in the equation (1), we get

$$17\left(x'+1\right)^2+18\left(x'+1\right)\left(y'-1\right)-7\left(y'-1\right)^2-16\left(x'+1\right)-32\left(y'-1\right)+18=0$$

$$\Rightarrow 17\left(x'^2+2x'+1\right)+18\left(x'y'-x'+y'-1\right)-7\left(y'^2-2y'+1\right)-16\left(x'+1\right)-32\left(y'-1\right)+18=0$$

$$\Rightarrow 17x'^2+34x'+17+18x'y'-18x'+18y'-18-7y'^2+14y'-7-16x'-16-32y'+32+18=0$$

$$\Rightarrow 17x'^2+18x'y'-7y'^2+26=0 \quad \cdots\cdots \quad (2)$$

Writing x for x' and y for y' in the equation (2), we obtain

$17x^2+18xy-7y^2+26=0$ which is required equation referred to the new axes.

Question: Transform to parallel axes through the point $(-2,3)$ of the following equation

$2x^2+4xy+5y^2-4x-22y+7=0$

Solution: The given equation is $2x^2+4xy+5y^2-4x-22y+7=0 \quad \cdots\cdots \quad (1)$

Putting $x=x'-2$ and $y=y'+3$ in the equation (1), we get

$$2(x'-2)^2+4(x'-2)(y'+3)+5(y'+3)^2-4(x'-2)-22(y'+3)+7=0$$

$$\Rightarrow 2\left(x'^2-4x'+4\right)+4\left(x'y'+3x'-2y'-6\right)+5\left(y'^2-6y'+9\right)-4\left(x'-2\right)-22\left(y'+3\right)+7=0$$

$$\Rightarrow 2x'^2-8x'+8+4x'y'+12x'-8y'-24+5y'^2-30y'+45-4x'+8-22y'-66+7=0$$

$$\Rightarrow 2x'^2+4x'y'+5y'^2-22=0 \quad \cdots\cdots \quad (2)$$

Writing x for x' and y for y' in the equation (2), we obtain

$2x^2+4xy+5y^2-22=0$ which is required equation referred to the new axes.

Question: Transform to axes inclined at 45^0 to the original axes the following equation

$$2x^2+4xy+5y^2-22=0$$

Solution: The given equation is $2x^2+4xy+5y^2-22=0 \quad \cdots\cdots \quad (1)$

By the theorem we have $x=x'\cos\theta-y'\sin\theta$ and $y=x'\sin\theta+y'\cos\theta$ where $\theta=45^0$ is given.

Therefore $x=x'\cos 45^0-y'\sin 45^0=\frac{1}{\sqrt{2}}(x'-y')$ and $y=x'\sin 45^0+y'\cos 45^0=\frac{1}{\sqrt{2}}(x'+y')$

Putting the value of x and y in the equation (1), we get

$$2\tfrac{1}{2}(x'-y')^2+4\tfrac{1}{\sqrt{2}}(x'-y')\tfrac{1}{\sqrt{2}}(x'+y')+5\tfrac{1}{2}(x'+y')^2-22=0$$
$$\Rightarrow (x'^2-2x'y'+y'^2)+2(x'+y')(x'-y')+\tfrac{5}{2}(x'^2+2x'y'+y'^2)-22=0$$
$$\Rightarrow x'^2-2x'y'+y'^2+2x'^2-2y'^2+\tfrac{5}{2}x'^2+5x'y'+\tfrac{5}{2}y'^2-22=0$$
$$\Rightarrow (1+2+\tfrac{5}{2})x'^2+3x'y'+(1-2+\tfrac{5}{2})y'^2-22=0$$
$$\Rightarrow \tfrac{11}{2}x'^2+3x'y'+\tfrac{3}{2}y'^2-22=0$$
$$\Rightarrow 11x'^2+6x'y'+3y'^2-44=0 \quad \cdots\cdots \quad (2)$$

Writing x for x' and y for y' in the equation (2), we obtain

$\therefore 11x^2+6xy+3y^2-44=0$ which is the required transformed equation.

Question: Transform to axes inclined at 45^0 to the original axes the following equation

$$3x^2+2xy+3y^2-1=0$$

Solution: The given equation is $3x^2+2xy+3y^2-1=0 \quad \cdots\cdots \quad (1)$

By the theorem we have $x=x'\cos\theta-y'\sin\theta$ and $y=x'\sin\theta+y'\cos\theta$ where $\theta=45^0$ is given.

Therefore $x=x'\cos 45^0-y'\sin 45^0=\frac{1}{\sqrt{2}}(x'-y')$ and $y=x'\sin 45^0+y'\cos 45^0=\frac{1}{\sqrt{2}}(x'+y')$

Putting the value of x and y in the equation (1), we get

$$3\tfrac{1}{2}(x'-y')^2+2\tfrac{1}{\sqrt{2}}(x'-y')\tfrac{1}{\sqrt{2}}(x'+y')+3\tfrac{1}{2}(x'+y')^2-1=0$$
$$\Rightarrow 3(x'^2-2x'y'+y'^2)+2(x'+y')(x'-y')+3(x'^2+2x'y'+y'^2)-2=0$$
$$\Rightarrow 3x'^2-6x'y'+3y'^2+2x'^2-2y'^2+3x'^2+6x'y'+3y'^2-2=0$$
$$\Rightarrow 8x'^2+4y'^2-2=0$$
$$\Rightarrow 4x'^2+2y'^2-1=0 \quad \cdots\cdots \quad (2)$$

Writing x for x' and y for y' in the equation (2), we obtain

$\therefore 4x^2+2y^2-1=0$ which is the required transformed equation.

Week 9:
Topics: 2nd degree general equation Page (73-77)

Question: Find the value of λ so that the following equation may represent pair of straight lines

$$6x^2+2\lambda xy+12y^2+22x+31y+20=0$$

Solution: The given equation is $6x^2+2\lambda xy+12y^2+22x+31y+20=0 \quad \cdots\cdots \quad (1)$

Comparing (1) with the general equation second degree, we have

$$a=6,\ h=\lambda,\ b=12,\ g=11,\ f=\frac{31}{2},\ c=20.$$

Now the condition for a pair of straight line is $\Delta=abc+2fgh-af^2-bg^2-ch^2=0$

$$\Rightarrow 6\times12\times20+2\times\tfrac{31}{2}\times11\times\lambda-6\left(\tfrac{31}{2}\right)^2-12\times(11)^2-20\lambda^2=0$$

$$\Rightarrow 1440+341\lambda-\tfrac{2883}{2}-1452-20\lambda^2=0$$

$$\Rightarrow 40\lambda^2-682\lambda+2907=0$$

$$\therefore\quad \lambda=\frac{682\pm\sqrt{(682)^2-4\times40\times2907}}{2\times40}=\frac{682\pm2}{80}=\frac{171}{20},\frac{17}{2}$$

Question: Find the value of λ so that the following equation may represent a pair of straight lines

$$6x^2+11xy-10y^2+x+31y+\lambda=0$$

Solution: The given equation is $6x^2+11xy-10y^2+x+31y+\lambda=0 \quad \cdots\cdots \quad (1)$

Comparing (1) with the general equation second degree, we have

$$a=6,\ h=\frac{11}{2},\ b=-10,\ g=\frac{1}{2},\ f=\frac{31}{2},\ c=\lambda.$$

Now the condition for a pair of straight line is $\Delta=abc+2fgh-af^2-bg^2-ch^2=0$

$$\Rightarrow 6\times(-10)\times\lambda+2\times\tfrac{31}{2}\times\tfrac{1}{2}\times\tfrac{11}{2}-6\left(\tfrac{31}{2}\right)^2-(-10)\times\left(\tfrac{1}{2}\right)^2-20\left(\tfrac{11}{2}\right)^2=0$$
$$\Rightarrow -60\lambda+\tfrac{341}{4}-\tfrac{2883}{2}+\tfrac{5}{2}-605=0$$
$$\Rightarrow -60\lambda-\tfrac{7835}{4}=0$$
$$\therefore\quad \lambda=\tfrac{-1567}{48}$$

Question: Prove that the equation $x^2-5xy+4y^2+x+2y-2=0$ represents a pair of straight lines. Also find their point of intersection and the angle between them.

Solution: The given equation is $x^2-5xy+4y^2+x+2y-2=0 \quad \cdots\cdots \quad (1)$

Comparing (1) with the general equation second degree, we have

$$a=1,\, h=-\frac{5}{2},\, b=4,\, g=\frac{1}{2},\quad f=1,\quad c=-2 .$$

Now $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$$=1\times4\times(-2)+2\times1\times\tfrac{1}{2}\times\left(-\tfrac{5}{2}\right)-1\times1^2-4\left(\tfrac{1}{2}\right)^2-(-2)\left(-\tfrac{5}{2}\right)^2=-8-\tfrac{5}{2}-1-1+\tfrac{25}{2}=0$$

Thus the given equation represents a pair of straight lines

Let $S(x,y)=x^2-5xy+4y^2+x+2y-2=0$, then we have

$S(\alpha,\beta)=\alpha^2-5\alpha\beta+4\beta^2+\alpha+2\beta-2=0$

Let (α,β) be the point of intersection of the given straight lines. Then we have

$$\frac{\partial S(\alpha,\beta)}{\partial\alpha}=2\alpha-5\beta+1=0 \quad\cdots\cdots\quad (2) \quad \textit{and} \quad \frac{\partial S(\alpha,\beta)}{\partial\beta}=-5\alpha+8\beta+2=0 \quad\cdots\cdots\quad (3)$$

Solving (2) and (3) we obtain $\alpha=2$ and $\beta=1$.

Thus the point of intersection is $(2,1)$.

Let θ be the angle between the two lines, then we have

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\frac{25}{4}-1\times4}}{1+4}=\frac{3}{5}\quad \therefore\theta=\tan^{-1}\left(\tfrac{3}{5}\right)$$

Question: Prove that the equation $2x^2+5xy+3y^2-9x-11y+10=0$ represents a pair of straight lines. Also find their point of intersection and the angle between them.

Solution: The given equation is $2x^2+5xy+3y^2-9x-11y+10=0 \quad \cdots\cdots \quad (1)$

Comparing (1) with the general equation second degree, we have

$$a=2,\ h=\frac{5}{2},\ b=3,\ g=-\frac{9}{2},\ \ f=-\frac{11}{2},\ \ c=10 .$$

Now $\Delta = abc+2fgh-af^2-bg^2-ch^2$

$$=2\times3\times10+2\times\left(-\tfrac{11}{2}\right)\times\left(-\tfrac{9}{2}\right)\times\left(\tfrac{5}{2}\right)-2\times\left(-\tfrac{11}{2}\right)^2-3\left(-\tfrac{9}{2}\right)^2-10\left(\tfrac{5}{2}\right)^2$$
$$=60+\tfrac{495}{4}-\tfrac{242}{4}-\tfrac{243}{4}-\tfrac{250}{4} \quad =\tfrac{240+495-242-243-250}{4} \quad =0$$

Thus the given equation represents a pair of straight lines

Let $S(x,y)=2x^2+5xy+3y^2-9x-11y+10=0$, then we have

$S(\alpha,\beta)=2\alpha^2+5\alpha\beta+3\beta^2-9\alpha-11\beta+10=0$

Let (α,β) be the point of intersection of the given straight lines. Then we have

$$\frac{\partial S(\alpha,\beta)}{\partial\alpha}=4\alpha+5\beta-9=0 \quad\cdots\cdots\quad (2) \quad and \quad \frac{\partial S(\alpha,\beta)}{\partial\beta}=5\alpha+6\beta-11=0 \quad\cdots\cdots\quad (3)$$

Solving (2) and (3) we obtain $\alpha=1$ and $\beta=1$.

Thus the point of intersection is $(1,1)$.

Let θ be the angle between the two lines, then we have

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\frac{25}{4}-2\times3}}{2+3}=\frac{1}{5} \qquad \therefore \theta=\tan^{-1}\left(\tfrac{1}{5}\right)$$

Question: Prove that the straight lines represented by the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

will be equidistant from the origin if $f^4-g^4=c\left(bf^2-ag^2\right)$

Solution: Let the straight lines represented by the given equation be

$l\,x+my+n=0$ and $l'\,x+m'y+n'=0 \quad \cdots\cdots \quad (1)$

Then $(l\,x+my+n)(l'\,x+m'y+n')=ax^2+2hxy+by^2+2gx+2\,f\,y+c$

$$\Rightarrow ll'\,x^2+ml'xy+nl'x+lm'xy+mm'y^2+nm'y+ln'x+mn'y+nn'=ax^2+2hxy+by^2+2gx+2\,f\,y+c$$
$$\Rightarrow ll'x^2+(lm'+ml')xy+mm'y^2+(ln'+nl')\,x+(mn'+nm')y+nn'=ax^2+2hxy+by^2+2gx+2\,f\,y+c$$

which is an identity.

Comparing the coefficients of both sides, we get

$$ll'=a,\quad (lm'+ml')=2h,\quad mm'=b,\quad (ln'+nl')=2g,\quad (mn'+nm')=2\,f\,,\quad nn'=c \quad \cdots\cdots \quad (2)$$

Since the lines are equidistance from the origin, we have

$$\frac{n}{\sqrt{l^2+m^2}}=\frac{n'}{\sqrt{l'^2+m'^2}}$$

Squaring both sides, we get

$$\frac{n^2}{l^2+m^2}=\frac{n'^2}{l'^2+m'^2}$$
$$\Rightarrow n^2\left(l'^2+m'^2\right)=n'^2\left(l^2+m^2\right)$$
$$\Rightarrow n^2l'^2-l^2n'^2=m^2n'^2-n^2m'^2$$
$$\Rightarrow (nl'+n'l)(nl'-n'l)=(mn'+nm')(mn'-nm')$$

Again squaring both sides, we have

$$(nl'+n'l)^2(nl'-n'l)^2=(mn'+nm')^2(mn'-nm')^2$$
$$\Rightarrow (nl'+n'l)^2\left\{(nl'+n'l)^2-4ll'nn'\right\}=(mn'+nm')^2\left\{(mn'+nm')^2-4mm'nn'\right\}$$

Substituting the values from equation (2), we obtain

$$4g^2\left(4g^2-4ac\right)=4\,f^{\,2}\left(4\,f^{\,2}-4bc\right)$$
$$\Rightarrow g^2\left(g^2-ac\right)=f^{\,2}\left(f^{\,2}-bc\right)$$
$$\Rightarrow g^4-g^2ac=f^{\,4}-f^{\,2}bc$$
$$\therefore\quad f^{\,4}-g^4=c\left(bf^{\,2}-ag^2\right) \qquad (Proved)$$

Week 10
Topics: Plane
Pages (77-80)

Question: Find the equation of the plane through the three points $(-1,0,1)$, $(-1,4,2)$ and $(2,4,1)$
.

Solution: The equation of the required plane is $\begin{vmatrix} x & y & z & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 4 & 2 & 1 \\ 2 & 4 & 1 & 1 \end{vmatrix} = 0$

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Expanding with respect to the forth column, we get $\Rightarrow \begin{vmatrix} x+1 & y & z-1 \\ 0 & 4 & 1 \\ -3 & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow (x+1)(4-0) - y(0+3) + (z-1)(0+12) = 0$

$\Rightarrow 4x + 4 - 3y + 12z - 12 = 0$

$\therefore 4x - 3y + 12z - 8 = 0$ which is the required equation of the plane.

Question: Find the equation of the plane through the three points $(2,2,-1)$, $(3,4,2)$ and $(7,0,6)$.

Solution: The equation of the required plane is $\begin{vmatrix} x & y & z & 1 \\ 2 & 2 & -1 & 1 \\ 3 & 4 & 2 & 1 \\ 7 & 0 & 6 & 1 \end{vmatrix} = 0$

Using row operations, we have $\begin{vmatrix} x-2 & y-2 & z+1 & 0 \\ 1 & 2 & 3 & 0 \\ 4 & -4 & 4 & 0 \\ 7 & 0 & 6 & 1 \end{vmatrix} = 0$

Expanding with respect to the forth column, we get $\Rightarrow \begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 4 & -4 & 4 \end{vmatrix} = 0$

$\Rightarrow (x-2)(8+12)-(y-2)(4-12)+(z+1)(-4-8)=0$

$\Rightarrow 20(x-2)+8(y-2)-12(z+1)=0$

$\Rightarrow 5(x-2)+2(y-2)-3(z+1)=0$

$\Rightarrow 5x-10+2y-4-3z-3=0$

$\therefore 5x+2y-3z-17=0$ which is the required equation of the plane.

Question: Find the equation of the plane through the points $(2,2,1)$ and $(9,3,6)$ and is perpendicular to the plane $2x+6y+6z-9=0$

Solution: The equation of the plane through the points $(2,2,1)$ is

$a(x-2)+b(y-2)+c(z-1)=0 \quad \cdots\cdots \quad (1)$

Since (1) passes through the point $(9,3,6)$, we get $7a+b+5c=0 \quad \cdots\cdots \quad (2)$

Again the plane (1) is perpendicular to the plane $2x+6y+6z-9=0$.

So we have $2a+6b+6c=0 \quad \cdots\cdots \quad (3)$

Now eliminating a, b, c from equation (1), (2) and (3) we obtain

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(6-30)+(y-2)(10-42)+(z-1)(42-2)=0$

$\Rightarrow -24(x-2)-32(y-2)+40(z-1)=0$

$\Rightarrow 3(x-2)+4(y-2)-5(z-1)=0$

$\therefore \quad 3x+4y-5z-9=0$ which is the required equation of the plane.

Question: Find the equation of a plane which passes through the intersection of the planes $7x-4y+7z+16=0$ and $4x+3y-2z+3=0$ and parallel to the plane $3x-7y+9z+5=0$

Solution: The equation of a plane which passes through the intersection of the planes $7x-4y+7z+16=0$ and $4x+3y-2z+3=0$ is

$7x-4y+7z+16+k(4x+3y-2z+3)=0 \quad \cdots\cdots \quad (1)$

Or $(4k+7)x+(3k-4)y+(-2k+7)z+3k+16=0 \quad \cdots\cdots \quad (2)$

Since (2) is parallel to the plane $3x-7y+9z+5=0$, we get $\dfrac{4k+7}{3}=\dfrac{3k-4}{-7}=\dfrac{-2k+7}{9}$

Taking first two equality we get $\dfrac{4k+7}{3}=\dfrac{3k-4}{-7} \Rightarrow -28k-49=9k-12 \Rightarrow -37k=37 \therefore k=-1$

Substituting the value of k in (1), we obtain

$7x-4y+7z+16-4x-3y+2z-3=0$

$\therefore \quad 3x-7y+9z+13=0$ which is the required equation of the plane.

Question: Find the equation of a plane which passes through the intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ and perpendicular to the plane $5x+3y+6z+8=0$

Solution: The equation of a plane which passes through the intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is $x+2y+3z-4+k(2x+y-z+5)=0 \quad \cdots\cdots \quad (1)$

$(2k+1)x+(k+2)y+(-k+3)z+5k-4=0 \quad \cdots\cdots \quad (2)$

Since (2) is perpendicular to the plane $5x+3y+6z+8=0$, we get

$5(2k+1)+3(k+2)+6(-k+3)=0 \quad \Rightarrow 10k+5+3k+6-6k+18=0 \Rightarrow 7k+29=0 \therefore k=-\dfrac{29}{7}$

Substituting the value of k in (1), we obtain

$$x+2y+3z-4-\frac{29}{7}(2x+y-z+5)=0$$

$$\Rightarrow 7x+14y+21z-28-58x-29y+29z-145=0$$

$$\Rightarrow -51x-15y+50z-173=0$$

$\therefore\ 51x+15y-50z+173=0$ which is the required equation of the plane.

Week 11

Topics: Direction cosines

Pages (80-85)

Question: If l, m, n be the direction cosines of a line then show that $l^2+m^2+n^2=1$

Solution: Let AB be the line which direction cosines are l, m, n. Through O draw a line OP parallel to AB, then the direction cosines of OP are also l, m, n.

Let $OP=r$ and P be the point (x, y, z), then we have $x=lr,\ y=mr$ and $z=nr$.

Squaring and adding we get

$$x^2+y^2+z^2=r^2\left(l^2+m^2+n^2\right)$$

$\Rightarrow r^2=r^2\left(l^2+m^2+n^2\right)$, since $r^2=OP^2=x^2+y^2+z^2$

$\therefore\ l^2+m^2+n^2=1$

Question: Find the angle between the lines whose direction ratios are $1, 1, 2$ and $\sqrt{3}-1, -\sqrt{3}-1, 4$.

Solution: Let l_1, m_1, n_1 be the direction cosines of the first line and l_2, m_2, n_2 be those of the second line. Then we have

$$\frac{l_1}{1}=\frac{m_1}{1}=\frac{n_1}{2}=\frac{\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{1^2+1^2+2^2}}=\frac{1}{\sqrt{6}} \quad \therefore l_1=\frac{1}{\sqrt{6}}, m_1=\frac{1}{\sqrt{6}}, n_1=\frac{2}{\sqrt{6}}$$

Again, $\dfrac{l_2}{\sqrt{3}-1}=\dfrac{m_2}{-\sqrt{3}-1}=\dfrac{n_2}{4}=\dfrac{\sqrt{l_2^2+m_2^2+n_2^2}}{\sqrt{\left(\sqrt{3}-1\right)^2+\left(-\sqrt{3}-1\right)^2+4^2}}=\dfrac{1}{\sqrt{24}}=\dfrac{1}{2\sqrt{6}}$

$$\therefore l_2=\frac{\sqrt{3}-1}{2\sqrt{6}}, m_2=\frac{-\sqrt{3}-1}{2\sqrt{6}}, n_2=\frac{2}{\sqrt{6}}$$

Now if θ be the angle between the lines, then

$$\cos\theta=l_1l_2+m_1m_2+n_1n_2$$

$$=\frac{1}{\sqrt{6}}\cdot\frac{\sqrt{3}-1}{2\sqrt{6}}+\frac{1}{\sqrt{6}}\cdot\frac{-\sqrt{3}-1}{2\sqrt{6}}+\frac{2}{\sqrt{6}}\cdot\frac{2}{\sqrt{6}}$$

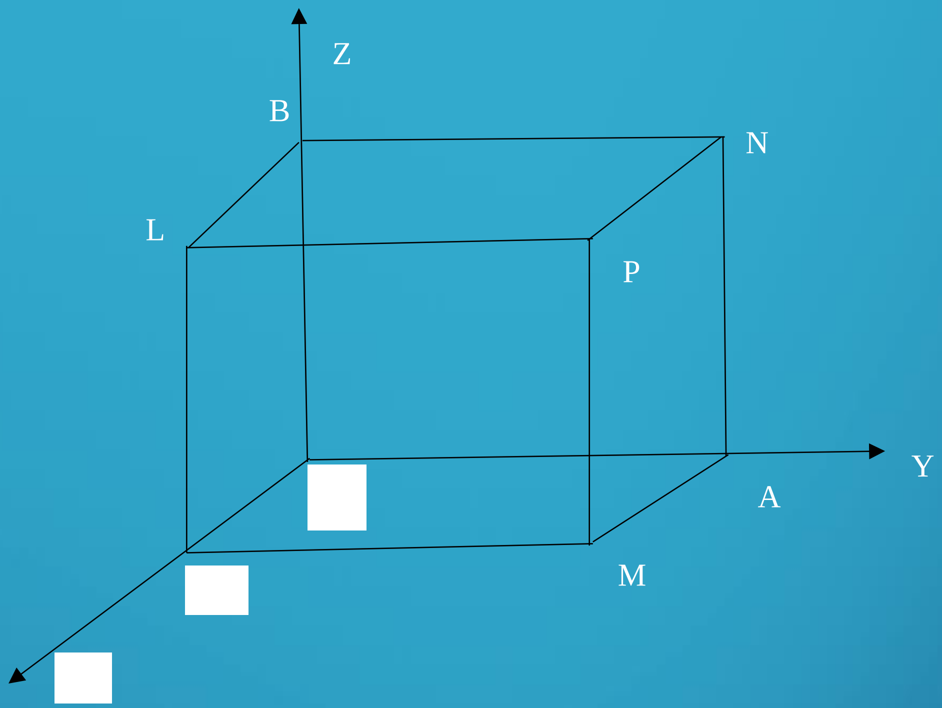
$$=\frac{\sqrt{3}-1}{12}+\frac{-\sqrt{3}-1}{12}+\frac{4}{6}=\frac{\sqrt{3}-1-\sqrt{3}-1+8}{12}=\frac{6}{12}=\frac{1}{2}=\cos\frac{\pi}{3}$$

$$\therefore \theta=\frac{\pi}{3}$$

Question: A line makes angles $\alpha,\beta,\gamma,\delta$ with four diagonals of a cube. Prove that

$(i)\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\dfrac{4}{3} \quad (ii)\quad \sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta=\dfrac{8}{3}$

Solution:



Let OCMANBLP be a cube whose each edge is a. The coordinates of the vertices are respectively, $O(0,0,0)$, $A(a,0,0)$, $B(0,a,0)$, $C(0,0,a)$, $P(a,a,a)$, $L(0,a,a)$, $M(a,0,a)$ and $N(a,a,0)$. Now $OP=\sqrt{(a-0)^2+(a-0)^2+(a-0)^2}=\sqrt{3a^2}=\sqrt{3}a$. Here the four diagonals OP, AL, BM and CN of the cube are equal. That is $OP=AL=BM=CN=\sqrt{3}a$.

The direction cosine of the diagonal OP are $\frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}$ or $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

The direction cosine of the diagonal AL are $\frac{0-a}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}$ or $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

The direction cosine of the diagonal BM are $\frac{a-0}{a\sqrt{3}}, \frac{0-a}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}$ or $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

The direction cosine of the diagonal CN are $\frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}, \frac{0-a}{a\sqrt{3}}$ or $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

Let l,m,n be the direction cosines of the line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals OP, AL, BM and CN respectively. Then

$$\cos\alpha = l\left(\frac{1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)=\frac{l+m+n}{\sqrt{3}} \quad \cdots\cdots \quad (1)$$

$$\cos\beta = l\left(\frac{-1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)=\frac{-l+m+n}{\sqrt{3}} \quad \cdots\cdots \quad (2)$$

$$\cos\gamma = l\left(\frac{1}{\sqrt{3}}\right)+m\left(\frac{-1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)=\frac{l-m+n}{\sqrt{3}} \quad \cdots\cdots \quad (3)$$

$$\cos\delta = l\left(\frac{1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{-1}{\sqrt{3}}\right)=\frac{l+m-n}{\sqrt{3}} \quad \cdots\cdots \quad (4)$$

Now squaring and adding (1), (2), (3) and (4), we obtain

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{1}{3}\left[(l+m+n)^2+(-l+m+n)^2+(l-m+n)^2+(l+m-n)^2\right]$$

$$=\frac{4}{3}\left(l^2+m^2+n^2\right)=\frac{4}{3} \qquad \text{since} \quad l^2+m^2+n^2=1$$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{4}{3}$$

(ii) We have $\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{4}{3}$

$$or, 1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma+1-\sin^2\delta = \frac{4}{3}$$

$$or, \sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta = 4-\frac{4}{3}=\frac{8}{3}$$

$$\therefore \quad \sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta = \frac{8}{3}$$

Question: Prove that the straight lines whose direction cosines are given by the relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel if $\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$.

Solution: The given relations are $al+bm+cn=0 \quad \cdots\cdots(1)$ and $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0 \quad \cdots\cdots(2)$

Proof of first part:

Eliminating n between (1) and (2), we get

$$(fm+gl)\left(\frac{-al-bm}{c}\right)+hlm=0$$

$$\Rightarrow alfm+agl^2+bfm^2+bglm-hclm=0$$

$$\Rightarrow agl^2+(af+bg-ch)lm+bfm^2=0$$

$$\Rightarrow ag\left(\tfrac{l}{m}\right)^2+(af+bg-ch)\tfrac{l}{m}+bf=0 \quad \cdots\cdots \quad (3)$$

which is a quadratic equation in $\dfrac{l}{m}$.

Let $\dfrac{l_1}{m_1}$ and $\dfrac{l_2}{m_2}$ be the roots of the equation (3), then

product of roots, $\dfrac{l_1 l_2}{m_1 m_2}=\dfrac{bf}{ag}=\dfrac{\frac{f}{a}}{\frac{g}{b}}$ or, $\dfrac{l_1 l_2}{\frac{f}{a}}=\dfrac{m_1 m_2}{\frac{g}{b}} \quad \cdots\cdots(4)$

Again eliminating l from (1) and (2), we obtain $\dfrac{m_1 m_2}{\frac{g}{b}}=\dfrac{n_1 n_2}{\frac{h}{c}} \quad \cdots\cdots(5)$

Combining (4) and (5), we get $\dfrac{l_1 l_2}{\frac{f}{a}}=\dfrac{m_1 m_2}{\frac{g}{b}}=\dfrac{n_1 n_2}{\frac{h}{c}}=k\,(say)$

Therefore $l_1 l_2 = k\frac{f}{a}, \quad m_1 m_2 = k\frac{g}{b}, \quad n_1 n_2 = k\frac{h}{c}$

Now two lines will be perpendicular if

$$l_1 l_2+m_1 m_2+n_1 n_2=0 \quad i.e \quad k\frac{f}{a}+k\frac{g}{b}+k\frac{h}{c}=0 \quad \therefore \frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$$

Proof of second part:

The two will be parallel if $\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2} \quad \Rightarrow \dfrac{l_1}{m_1}=\dfrac{l_2}{m_2}$

That is, the two roots of the equation (3) are equal, the condition for which is

$$\left(af+bg-ch\right)^2-4\cdot ag\cdot bf=0$$

$$\Rightarrow \left(af+bg-ch\right)^2=4\cdot ag\cdot bf$$

$$\Rightarrow af+bg-ch=\pm 2\sqrt{ag\cdot bf}$$

$$\Rightarrow af\pm 2\sqrt{af\cdot bg}+bg=ch$$

$$\Rightarrow \left(\sqrt{af}\pm\sqrt{bg}\right)^2=ch$$

$$\Rightarrow \sqrt{af}\pm\sqrt{bg}=\pm\sqrt{ch}$$

$$\therefore \sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$$

Week 12:
Topics: Parabola, Shortest Distance
Pages (85-94)

Question: Reduce the conic $4x^2-4xy+y^2-8x-6y+5=0$ to its standard form and also find its properties.

Solution: The given equation of the conic is $4x^2-4xy+y^2-8x-6y+5=0 \quad \cdots\cdots \quad (1)$

Comparing the equation (1) with second degree general equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get $a=4,\ h=-2,\ b=1,\ g=-4,\ f=-3,\ c=5$

$\therefore \Delta=abc+2fgh-af^2-bg^2-ch^2=20-48-36-16-20=-100\neq 0$ and $ab-h^2=4-4=0$

Hence the given equation represents a parabola.

Now the equation (1) can be written as $\left(2x-y\right)^2=8x+6y-5 \quad \cdots\cdots \quad (2)$

Or, $\left(2x-y+\lambda\right)^2=\left(8+4\lambda\right)x+\left(6-2\lambda\right)y+\lambda^2-5 \quad \cdots\cdots \quad (3)$

The two lines $2x-y+\lambda=0$ and $(8+4\lambda)x+(6-2\lambda)y+\lambda^2-5=0$ will be perpendicular if $a_1a_2+b_1b_2=0$, that is $2(8+4\lambda)+(-1)(6-2\lambda)=0 \Rightarrow 10\lambda+10=0 \quad \therefore \lambda=-1$

Putting $\lambda=-1$ in (3), we get $(2x-y-1)^2=4x+8y-4=4(x+2y-1)$ which can be written as

$$\left(\frac{2x-y-1}{\sqrt{2^2+(-1)^2}}\right)^2\times 5=4\sqrt{5}\left(\frac{x+2y-1}{\sqrt{1^2+2^2}}\right)$$

$$\left(\frac{2x-y-1}{\sqrt{2^2+(-1)^2}}\right)^2=\frac{4}{\sqrt{5}}\left(\frac{x+2y-1}{\sqrt{1^2+2^2}}\right)$$

Which is of the form $Y^2=4AX \quad \cdots\cdots \quad (4)$

Where $4A=\frac{4}{\sqrt{5}}$, $Y=\frac{2x-y-1}{\sqrt{5}}$, $X=\frac{x+2y-1}{\sqrt{5}}$

Equation (4) is the standard form of the **parabola**.

(i) Length of the lotus rectum $=4A=\frac{4}{\sqrt{5}}$

(ii) Axis is $Y=0$, $i.e,$ $2x-y-1=0$

(iii) Vertex is $X=0$, $Y=0$, $i.e,$ $x+2y-1=0$ and $2x-y-1=0$.

Solving these above equations, we get $\left(\frac{3}{5}, \frac{1}{5}\right)$

(iv) Focus is $(A,0)$, $i.e,$ $X=A$, $Y=0$

Or, $\frac{x+2y-1}{\sqrt{5}}=\frac{1}{\sqrt{5}}$, $\frac{2x-y-1}{\sqrt{5}}=0$

Or, $x+2y-2=0$, $2x-y-1=0$

Solving the above two linear equations, we get the coordinates of the focus as $\left(\frac{4}{5}, \frac{3}{5}\right)$

(v) Equation of the directrix is $X=-A$, i.e, $\frac{x+2y-1}{\sqrt{5}}=-\frac{1}{\sqrt{5}} \quad \therefore x+2y=0$

(vi) Equation of the latus rectum is $X=A$, i.e, $\frac{x+2y-1}{\sqrt{5}}=\frac{1}{\sqrt{5}} \quad \therefore x+2y-2=0$

(vii) Foot of the directrix is $(-A,0)$, *i.e*, $X=-A,\ Y=0$

Or, $\dfrac{x+2y-1}{\sqrt{5}}=-\dfrac{1}{\sqrt{5}},\quad \dfrac{2x-y-1}{\sqrt{5}}=0$

Or, $x+2y=0,\quad 2x-y-1=0$

Solving the above two linear equations, we get the coordinates of the foot of the directrix as $\left(\frac{2}{5},-\frac{1}{5}\right)$

Question: Reduce the conic $x^2-4xy+4y^2+10x-8y+13=0$ to its standard form and also find its properties.

Solution: The given equation of the conic is $x^2-4xy+4y^2+10x-8y+13=0 \quad \cdots\cdots \quad (1)$

Comparing the equation (1) with second degree general equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get $a=1, h=-2, b=4, g=5, f=-4, c=13$

$\therefore \Delta = abc+2fgh-af^2-bg^2-ch^2=56+80-16-100-56=-36\neq 0$ and $ab-h^2=4-4=0$

Hence the given equation represents a parabola.

Now the equation (1) can be written as $(x-2y)^2=-10x+8y-13 \quad \cdots\cdots \quad (2)$

Or, $(x-2y+\lambda)^2=(-10+2\lambda)x+(8-4\lambda)y+\lambda^2-13 \quad \cdots\cdots \quad (3)$

The two lines $x-2y+\lambda=0$ and $(-10+2\lambda)x+(8-4\lambda)y+\lambda^2-13=0$ will be perpendicular if $a_1a_2+b_1b_2=0$, that is $(-10+2\lambda)+(-2)(8-4\lambda)=0 \Rightarrow 10\lambda-26=0 \quad \therefore \lambda=\dfrac{13}{5}$

Putting $\lambda=\dfrac{13}{5}$ in (3), we get $\left(x-2y+\frac{13}{5}\right)^2=-\dfrac{12}{5}\left(2x+y+\frac{13}{5}\right)$ which can be written as

$$\left(\frac{x-2y+\frac{13}{5}}{\sqrt{1^2+(-2)^2}}\right)^2\times 5=-\frac{12}{5}\sqrt{5}\left(\frac{2x+y+\frac{13}{5}}{\sqrt{2^2+1^2}}\right)$$

$$\left(\frac{x-2y+\frac{13}{5}}{\sqrt{5}}\right)^2=-\frac{12}{5\sqrt{5}}\left(\frac{2x+y+\frac{13}{5}}{\sqrt{5}}\right)$$

Which is of the form $Y^2=4AX \quad \cdots\cdots \quad (4)$

Where $4A=\frac{-12}{5\sqrt{5}}$, $Y=\frac{x-2y+\frac{13}{5}}{\sqrt{5}}$, $X=\frac{2x+y+\frac{13}{5}}{\sqrt{5}}$

Equation (4) is the standard form of the **parabola**.

(i) Length of the lotus rectum $=4A=\frac{-12}{5\sqrt{5}}$

(ii) Axis is $Y=0$, $i.e$, $x-2y+\frac{13}{5}=0$

(iii) Vertex is $X=0$, $Y=0$, $i.e$, $2x+y+\frac{13}{5}=0$ *and* $x-2y+\frac{13}{5}=0$.

Solving these above equations, we get $\left(\frac{39}{25},\frac{13}{25}\right)$

(iv) Focus is $(A,0)$, $i.e$, $X=A$, $Y=0$

Or, $\frac{2x+y+\frac{13}{5}}{\sqrt{5}}=\frac{-3}{5\sqrt{5}}$, $\frac{x-2y+\frac{13}{5}}{\sqrt{5}}=0$

Or, $2x+y+\frac{16}{5}=0$, $x-2y+\frac{13}{5}=0$

Solving the above two linear equations, we get the coordinates of the focus as $\left(-\frac{9}{5},\frac{2}{5}\right)$

(v) Equation of the directrix is $X=-A$, i.e, $\frac{2x+y+\frac{13}{5}}{\sqrt{5}}=\frac{3}{5\sqrt{5}}$ $\therefore 2x+y+2=0$

(vi) Equation of the latus rectum is $X=A$, i.e, $\frac{2x+y+\frac{13}{5}}{\sqrt{5}}=-\frac{3}{5\sqrt{5}}$ $\therefore 2x+y+\frac{16}{5}=0$

(vii) Foot of the directrix is $(-A,0)$, $i.e$, $X=-A$, $Y=0$

Or, $\frac{2x+y+\frac{13}{5}}{\sqrt{5}}=\frac{3}{5\sqrt{5}}$, $\frac{x-2y+\frac{13}{5}}{\sqrt{5}}=0$

Or, $2x+y+2=0$, $x-2y+\frac{13}{5}=0$

Solving the above two linear equations, we get the coordinates of the foot of the directrix as $\left(-\frac{33}{25},\frac{16}{25}\right)$

Question: Find the shortest distance and the equation of the shortest distance line between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$

Solution: Let l,m,n be the direction cosines of the S.D**,** then since S.D is perpendicular to both the lines, we have $2l+3m+4n=0$ and $3l+4m+5n=0$.

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9} \text{ or, } \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+(-2)^2+1^2}}=\frac{1}{\sqrt{6}}$$

Hence the direction cosines of S.D are $\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}$

A point on the first line is say $P(1,2,3)$ and a point on the second line is say $Q(2,4,5)$

Now length of S.D= Projection of PQ on the line S.D

$$\text{S.D}=(2-1)\frac{1}{\sqrt{6}}+(4-2)\frac{-2}{\sqrt{6}}+(5-3)\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6}}-\frac{4}{\sqrt{6}}+\frac{2}{\sqrt{6}}=-\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6}}(magnitude)$$

Equation of the shortest distance

The line of S.D is obtained by the intersection of

(i) a plane containing first line and parallel to S.D
(ii) a plane containing second line and parallel to S.D

The equation of the plane containing first line and S.D is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}=0, \text{ i.e, } 11x+2y-7z+6=0 \quad \cdots\cdots(1)$$

The equation of the plane containing second line and S.D is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 2 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix}=0, \text{ i.e, } 7x+y-5z+7=0 \quad \cdots\cdots(2)$$

Now (1) and (2) together give equation of S.D

Question: Find the shortest distance and the equation of the shortest distance line between the lines $\frac{x-1}{4}=\frac{y-2}{3}=\frac{z-1}{-5}$ and $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-4}{-4}$

Solution: Let l, m, n be the direction cosines of the S.D**,** then since S.D is perpendicular to both the lines, we have $4l+3m-5n=0$ and $2l+3m-4n=0$.

$$\frac{l}{-12+15}=\frac{m}{-10+16}=\frac{n}{12-6} \text{ or, } \frac{l}{3}=\frac{m}{6}=\frac{n}{6}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{3^2+6^2+6^2}}=\frac{1}{9}$$

Hence the direction cosines of S.D are $\frac{1}{3},\frac{2}{3},\frac{2}{3}$

A point on the first line is say $P(1,2,1)$ and a point on the second line is say $Q(-1,3,4)$

Now length of S.D= Projection of PQ on the line S.D

$$\text{S.D}=(-1-1)\frac{1}{3}+(3-2)\frac{2}{3}+(4-1)\frac{2}{3}=-\frac{2}{3}+\frac{2}{3}+2=2$$

Equation of the shortest distance

The line of S.D is obtained by the intersection of

(i) a plane containing first line and parallel to S.D
(ii) a plane containing second line and parallel to S.D

The equation of the plane containing first line and S.D is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ 4 & 3 & -5 \\ 3 & 6 & 6 \end{vmatrix}=0, \text{ i.e, } 16x-13y+5z+5=0 \quad \cdots\cdots(1)$$

The equation of the plane containing second line and S.D is

$$\begin{vmatrix} x+1 & y-3 & z-4 \\ 2 & 3 & -4 \\ 3 & 6 & 6 \end{vmatrix}=0, \text{ i.e, } 14x-8y+z+34=0 \quad \cdots\cdots(2)$$

Now (1) and (2) together give equation of S.D

Exercise: Find the shortest distance (S.D) and the equation of the shortest distance (S.D) line between the lines $\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

Question: Show that the lines $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ are coplanar.

Also find their common point and the equation of the equation of the plane in which they lie.

Solution: The given equations of the lines are $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}=r_1 \quad \cdots\cdots \quad (1)$

and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}=r_2 \quad \cdots\cdots \quad (2)$

Now any point on the line (1) is $(4r_1+5,\ 4r_1+7,\ -5r_1-3) \quad \cdots\cdots \quad (3)$

and any point on the line (2)is $(7r_2+8,\ r_2+4,\ 3r_2+5) \quad \cdots\cdots \quad (4)$

If the two lines meet in a point, then for some values of r_1 and r_2 the points given by (3)and (4)must be identical.

$$4r_1+5=7r_2+8,\ 4r_1+7=r_2+4,\ -5r_1-3=3r_2+5$$
$$\Rightarrow 4r_1-7r_2=3,\ 4r_1-r_2=-3,\ 5r_1+3r_2=-8$$

Solving the first and second of the above equations, we get $r_1=-1, \quad r_2=-1$ which also satisfy the third equation $5r_1+3r_2=-8$. Hence the given two lines intersect and so they are coplanar.

Putting the value of r_1 and r_2 in (3) and (4) respectively, we get $(1,3,2)$ and $(1,3,2)$.

Hence the required common point of intersection is $(1,3,2)$.

The equation of the plane in which these lines lie is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=0$$
$$\Rightarrow (x-5)(12+5)-(y-7)(35+12)+(z+3)(4-28)=0$$
$$\Rightarrow 17x-85-47y+329-24z-72=0$$
$$\therefore \quad 17x-47y-24z+172=0$$

Question: Show that the lines $3x+2y+z-5=0=x+y-2z-3$ and $2x-y-z-16=0=7x+10y-8z-15$ are mutually perpendicular.

Solution: The given equation of the two lines are $3x+2y+z-5=0=x+y-2z-3 \quad \cdots\cdots \quad (1)$

and $2x-y-z-16=0=7x+10y-8z-15 \quad \cdots\cdots \quad (2)$

Let l_1, m_1, n_1 be the direction cosines of (1), then we have

$3l_1+2m_1+n_1=0 \quad \cdots\cdots \quad (3)$

$l_1+m_1-2n_1=0 \quad \cdots\cdots \quad (4)$

Now from (3) and (4) by cross multiplication, we get $\dfrac{l_1}{-4-1}=\dfrac{m_1}{1+6}=\dfrac{n_1}{3-2}$

Or, $\dfrac{l_1}{-5}=\dfrac{m_1}{7}=\dfrac{n_1}{1}=\dfrac{\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{(-5)^2+7^2+1^2}}=\dfrac{1}{\sqrt{75}}=\dfrac{1}{5\sqrt{3}} \quad \therefore l_1=\dfrac{-1}{\sqrt{3}}, \quad m_1=\dfrac{7}{5\sqrt{3}}, \quad n_1=\dfrac{1}{5\sqrt{3}}$

Again let l_2, m_2, n_2 be the direction cosines of (2), then we have

$2l_2-m_2-n_2=0 \quad \cdots\cdots \quad (5)$

$7l_2+10m_2-8n_2=0 \quad \cdots\cdots \quad (6)$

Now from (5) and (6) by cross multiplication, we get $\dfrac{l_2}{8+10}=\dfrac{m_2}{-7+16}=\dfrac{n_2}{20+7}$

Or, $\dfrac{l_2}{2}=\dfrac{m_2}{1}=\dfrac{n_2}{3}=\dfrac{\sqrt{l_2^2+m_2^2+n_2^2}}{\sqrt{2^2+1^2+3^2}}=\dfrac{1}{\sqrt{14}} \quad \therefore l_2=\dfrac{2}{\sqrt{14}}, \quad m_2=\dfrac{1}{\sqrt{14}}, \quad n_2=\dfrac{3}{\sqrt{14}}$

Here $l_1l_2+m_1m_2+n_1n_2=\dfrac{-10+7+3}{5\sqrt{3}\cdot\sqrt{14}}=0$

Hence the given two lines are mutually perpendicular.

Exercise: Show that the lines $2x+3y+4z-5=0=3x+4y+5z-6$ and $x+2y-3z-3=0=2x-5y+3z+3$are mutually perpendicular.

Question: Show that the lines $2x+3y-4z=0=3x-4y+z-7$ and $5x-y-3z+12=0=x-7y+4z-6$ are parallel.

Solution: The given equation of the two lines are $2x+3y-4z=0=3x-4y+z-7 \quad \cdots\cdots \quad (1)$

and $5x-y-3z+12=0=x-7y+4z-6 \quad \cdots\cdots \quad (2)$

Let l_1, m_1, n_1 be the direction cosines of (1), then we have

$$2l_1+3m_1-4n_1=0 \quad \cdots\cdots \quad (3)$$
$$3l_1-4m_1+n_1=0 \quad \cdots\cdots \quad (4)$$

Now from (3) and (4) by cross multiplication, we get $\dfrac{l_1}{3-16}=\dfrac{m_1}{-12-2}=\dfrac{n_1}{-8-9}$

Or, $\dfrac{l_1}{13}=\dfrac{m_1}{14}=\dfrac{n_1}{17} \quad \cdots\cdots \quad (5)$

Again let l_2, m_2, n_2 be the direction cosines of (2), then we have

$$5l_2-m_2-3n_2=0 \quad \cdots\cdots \quad (6)$$
$$l_2-7m_2+5n_2=0 \quad \cdots\cdots \quad (7)$$

Now from (6) and (7) by cross multiplication, we get $\dfrac{l_2}{-5-21}=\dfrac{m_2}{-3-25}=\dfrac{n_2}{-35+1}$

Or, $\dfrac{l_2}{-26}=\dfrac{m_2}{-28}=\dfrac{n_2}{-34} \quad or, \quad \dfrac{l_2}{13}=\dfrac{m_2}{14}=\dfrac{n_2}{17} \quad \cdots\cdots \quad (8)$

Therefore from (5) and (8) it is clear that $\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$

Hence the given two lines are parallel.

Exercise: Show that the lines $x+y-z-5=0=9x-5y+z$ and $6x-8y+4z-3=0=x+8y-6z+7$ are parallel.

Week 13:

Topics: Vector

Pages (94-99)

Vector: A vector is an object that has both a magnitude and a direction. A vector can be denoted as A, **A**, $\vec{A}$, $\underline{A}$. Suppose, $\underline{A} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, be a vector in 3-D, where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors along X, Y and Z-axis. We can also write it as $\vec{A} = (a, b, c)$.

Magnitude of a vector:

If $\vec{A} = a\hat{\imath} + b\hat{\jmath} - c\hat{k}$ be a vector then the magnitude of $\vec{A}$ is denoted by A or $|\vec{A}|$ and is defined as follows:

$$A = |\vec{A}| = \sqrt{a^2 + b^2 + (-c)^2}$$

Example: If $\vec{B} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$, then the magnitude of $\vec{B}$ is,

$$B = |\vec{B}| = \sqrt{2^2 + 4^2 + (-5)^2} = \sqrt{4 + 16 + 25} = \sqrt{45}$$

Example: If $\vec{A} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$, $\vec{B} = 5\hat{\imath} - 4\hat{\jmath} + 3\hat{k}$, $\vec{C} = 2\hat{\imath} - 4j + 2\hat{k}$

Then $\vec{A} - 2\vec{B} + \vec{C} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k} - 2(5\hat{\imath} + 4\hat{\jmath} - 5\hat{k}) + 2\hat{\imath} - 4j - 2\hat{k}$

$= 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k} - 10\hat{\imath} - 8\hat{\jmath} + 10\hat{k} + 2\hat{\imath} - 4j - 2\hat{k}$

$= -4\hat{\imath} - 15\hat{\jmath} + 10\hat{k}$

And $|\vec{A} - 2\vec{B} + \vec{C}| = \sqrt{(-4)^2 + (-15)^2 + 10^2} = \sqrt{341}$.

Vector addition, subtraction and scalar multiplication of vectors:

Suppose, $\underline{A} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$, $\underline{B} = 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\underline{C} = 2\hat{\imath} - 4j - 2\hat{k}$ are three vectors. Find the magnitude of $\underline{A} + \underline{B} + \underline{C}$ and $(3\underline{B} - 5\underline{C})$.

Then, $\underline{A} + \underline{B} + \underline{C} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k} + 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k} + 2\hat{\imath} - 4j - 2\hat{k} = 11\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$

$|\underline{A}+\underline{B}+\underline{C}| = \sqrt{(11)^2+(-3)^2+(-5)^2} = \sqrt{155}$

$3\underline{B}-5\underline{C} = 3.(5\hat{\imath}+4\hat{\jmath}-5\hat{k}) - 5.(2\hat{\imath}-4j-2\hat{k}) = 15\hat{\imath}+12\hat{\jmath}-15\hat{k}-10\hat{\imath}+20\hat{\jmath}+10\hat{k}$

$$= 5\hat{\imath}+32\hat{\jmath}-5\hat{k}$$

The magnitude of $(3\underline{B}-5\underline{C})$ is,

$$|3\underline{B}-5\underline{C}| = \sqrt{5^2+32^2+(-5)^2} = \sqrt{25+1024+25} = \sqrt{1074}$$

<u>Vector product</u>: There are two types of vector product. (i) Dot product,(ii) Cross product

<u>Dot product</u>: If $\underline{A} = a\hat{\imath}+b\hat{\jmath}+c\hat{k}$ and $\underline{B} = h\hat{\imath}+l\hat{\jmath}+m\hat{k}$ are two vectors then the dot product of $\underline{A}$ *and* $\underline{B}$ *is denoted by* $\underline{A}.\underline{B}$ and is defined as follows:

$\underline{A}.\underline{B} = (a\hat{\imath}+b\hat{\jmath}+c\hat{k}).(h\hat{\imath}+l\hat{\jmath}+m\hat{k}) = ah+bl+cm$

Since, $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ *and* $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

If $\underline{A}$ *and* $\underline{B}$ are two vectors and θ be the angle between them then

$\underline{A}.\underline{B} = |\underline{A}||\underline{B}|\cos\theta$ or, $|\underline{A}||\underline{B}|\cos\theta = \underline{A}.\underline{B}$ *or,* $\cos\theta = \frac{A.B}{|A||B|}$ or, $\theta = \cos^{-1}\left(\frac{A.B}{|A||B|}\right)$

Example: Suppose, $\underline{A} = 4\hat{\imath}-3\hat{\jmath}+2\hat{k}$, $\underline{B} = 5\hat{\imath}+4\hat{\jmath}-5\hat{k}$ *and* $\underline{C} = 2\hat{\imath}-4j-2\hat{k}$

(i) Find the value of $(\underline{A}+\underline{B}).(\underline{B}-\underline{C})$

(ii) Show that $(\underline{A}-\underline{B}).\underline{C} = \underline{A}.\underline{C}-\underline{B}.\underline{C}$

(iii) Show that $(\underline{A}+\underline{B}).\underline{C} = \underline{A}.\underline{C}+\underline{B}.\underline{C}$

(iv) Find the angle between $\underline{A}$ *and* $\underline{B}$ and between $\underline{B}$ *and* $\underline{C}$

(v) Show that $(\underline{A}+\underline{B}).(\underline{A}-\underline{B}) = |\underline{A}|^2-|\underline{B}|^2$

Solution: (vi): $\underline{A}+\underline{B} = 9\hat{\imath}+\hat{\jmath}-3\hat{k}, (\underline{A}-\underline{B}) = -\hat{\imath}-7\hat{\jmath}+7\hat{k}$

L.H.S=$(\underline{A}+\underline{B}).(\underline{A}-\underline{B}) = (9\hat{\imath}+\hat{\jmath}-3\hat{k})(\cdot-\hat{\imath}-7\hat{\jmath}+7\hat{k}) = -9-7-21 = -37$

Again $|\underline{A}| = \sqrt{29}$ and $|\underline{B}| = \sqrt{5^2+4^2+(-5)^2} = \sqrt{66}$

R.H.S=$|\underline{A}|^2 - |\underline{B}|^2 = 29 - 66 = -37$

$\underline{A}.\underline{B} = (4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}) = 20 - 12 - 10 = 20 - 22 = -2$

Solution: (i) $\underline{A} + \underline{B} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k} + 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k} = 9\hat{\imath} + \hat{\jmath} - 3\hat{k}($

$\underline{B} - \underline{C}) = 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k} - 2\hat{\imath} + 4j + 2\hat{k} = 3\hat{\imath} + 8j - 3\hat{k}$

So, $(\underline{A} + \underline{B}).(\underline{B} - \underline{C}) = (9\hat{\imath} + \hat{\jmath} - 3\hat{k}).(3\hat{\imath} + 8j - 3\hat{k})$

=27+8+9=44

Solution (iv): Let θ be the angle between $\underline{A}$ *and* $\underline{B}$.

We know that, $\underline{A}.\underline{B} = |\underline{A}||\underline{B}| \cos \cos \theta$

r, $\theta = \cos^{-1}\left(\frac{\underline{A}.\underline{B}}{|\underline{A}||\underline{B}|}\right)$……………….(i)

Now, $\underline{A}.\underline{B} = -2$, $|\underline{A}| = \sqrt{29}$ & $|\underline{B}| = \sqrt{66}$

From (i) we get, $\theta = \cos^{-1}\left(\frac{-2}{\sqrt{29}.\sqrt{66}}\right)$

Week 14:

Topics: Dot and Cross product

Pages (96-99)

Cross product: If $\underline{A} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ and $\underline{B} = h\hat{\imath} + l\hat{\jmath} + m\hat{k}$ are two vectors then the cross product of $\underline{A}$ *and* $\underline{B}$ *is denoted by* $\underline{A} \times \underline{B}$ and is defined as follows:

$$\underline{A} \times \underline{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ h & l & m \end{vmatrix} = (bm - cl)\hat{\imath} - (am - ch)\hat{\jmath} + (al - bh)\hat{k}$$

Example: If $\underline{A} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ and $\underline{B} = 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ are two vectors then find $(\underline{A} \times \underline{B})$.

$$(\underline{A} \times \underline{B}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & -3 & 2 \\ 5 & 4 & -5 \end{vmatrix} = 7\hat{\imath} + 30\hat{\jmath} + 31\hat{k}$$

$(\underline{B} \times \underline{A}) = -7\hat{\imath} - 30\hat{\jmath} - 31\hat{k} = -(7\hat{\imath} + 30\hat{\jmath} + 31\hat{k}) = -(\underline{A} \times \underline{B})$

Again, If $\underline{A}$ *and* $\underline{B}$ are two vectors and θ be the angle between them then

$$\underline{A} \times \underline{B} = |\underline{A}||\underline{B}| \sin\theta$$

Unit vector: If $\underline{A}$ be any vector and $|\underline{A}|$ be the magnitude of $\underline{A}$ then the unit vector along $\underline{A}$ is denoted by $\hat{a}$ and is defined as follows $\hat{a} = \frac{\underline{A}}{|\underline{A}|}$; $|\hat{a}| = 1$.

The unit vector along $(\underline{A} \times \underline{B})$ is $= \frac{(\underline{A}\times\underline{B})}{|\underline{A}\times\underline{B}|}$

Question: If $\underline{A} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ *and* $\underline{B} = 5\hat{\imath} - 4\hat{\jmath} - 5\hat{k}$ then find the unit vector along $(\underline{A} + \underline{B})$.

Solution: Given, $\underline{A} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ *and* $\underline{B} = 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

Now $\underline{A} + \underline{B} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k} + 5\hat{\imath} + 4\hat{\jmath} - 5\hat{k} = 9\hat{\imath} + \hat{\jmath} - 3\hat{k}$

And $|\underline{A} + \underline{B}| = \sqrt{9^2 + 1^2 + (-3)^2} = \sqrt{81 + 1 + 9} = \sqrt{91}$

So, the unit vector along $(\underline{A} + \underline{B})$ is $= \frac{(\underline{A}+\underline{B})}{|\underline{A}+\underline{B}|} = \frac{9\hat{\imath}+\hat{\jmath}-3\hat{k}}{\sqrt{91}} = \frac{9}{\sqrt{91}}\hat{\imath} + \frac{1}{\sqrt{91}}\hat{\jmath} - \frac{3}{\sqrt{91}}\hat{k}$

Question: If $\underline{P} = 4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, $\underline{Q} = 7\hat{\imath} + 2\hat{\jmath} - 5\hat{k}$ *and* $\underline{R} = 3\hat{\imath} - 4\hat{\jmath} - \hat{k}$.

Then, find the unit vector along $(\underline{P} \times \underline{Q})$, $(\underline{P} + \underline{Q} - \underline{R})$, $(\underline{P} - 2\underline{Q} - 3\underline{R})$ and $(\underline{P} + \underline{Q} + \underline{R})$

Theorem: If $\underline{A}$ and $\underline{B}$ are two vectors and θ be the angle between them then show that $|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = |\underline{A}|^2 . |\underline{B}|^2$

Proof: We know that, If $\underline{A}$ *and* $\underline{B}$ are two vectors and θ be the angle between them then

$$\underline{A}.\underline{B} = |\underline{A}||\underline{B}| \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\& \quad \underline{A} \times \underline{B} = |\underline{A}||\underline{B}| \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Squaring and Adding (i) & (ii) we get,

$$(\underline{A}.\underline{B})^2 + (\underline{A} \times \underline{B})^2 = |\underline{A}|^2.|\underline{B}|^2 sin^2\theta + |\underline{A}|^2.|\underline{B}|^2 cos^2\theta$$

$$= |\underline{A}|^2.|\underline{B}|^2 (sin^2\theta + cos^2\theta)$$

Or $(\underline{A}.\underline{B})^2 + (\underline{A} \times \underline{B})^2 = |\underline{A}|^2.|\underline{B}|^2$ [Since $sin^2\theta + cos^2\theta = 1$]

$\therefore$ $|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = |\underline{A}|^2.|\underline{B}|^2$

Since in vector analysis $(\underline{A}.\underline{B})^2 = |\underline{A}.\underline{B}|^2$ & $(\underline{A} \times \underline{B})^2 = |\underline{A} \times \underline{B}|^2$.

So, $|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = |\underline{A}|^2.|\underline{B}|^2$ **(Showed)**

Question: If $\underline{A} = \hat{\imath} + 3\hat{\jmath} + \hat{k}$, $\underline{B} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$, prove that $|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = |\underline{A}|^2.|\underline{B}|^2$

Solution: Given, $\underline{A} = \hat{\imath} + 3\hat{\jmath} + \hat{k}, \underline{B} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

So, $|\underline{A}| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{1 + 9 + 1} = \sqrt{11}$

And $|\underline{B}| = \sqrt{2^2 + 4^2 + (-5)^2} = \sqrt{4 + 16 + 25} = \sqrt{45}$.

Also, $\underline{A}.\underline{B} = (\hat{\imath} + 3\hat{\jmath} + \hat{k}).(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}) = 2 + 12 - 5 = 9$

And $\underline{A} \times \underline{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 1 \\ 2 & 4 & -5 \end{vmatrix} = (-15 - 4)\hat{\imath} - (-5 - 2)\hat{\jmath} + (4 - 6)\hat{k} = -19\hat{\imath} + 7\hat{\jmath} - 2\hat{k}$

$|\underline{A} \times \underline{B}| = \sqrt{(-19)^2 + 7^2 + (-2)^2} = \sqrt{414}$

L.H.S=$|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = 81 + 414 = 495$ and R.H.S= $|\underline{A}|^2.|\underline{B}|^2$ = 11.45=495

Hence, $|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = |\underline{A}|^2.|\underline{B}|^2$ **(Proved)**

Exercise: If $\underline{A} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$, $\underline{B} = \hat{\imath} - 4\hat{\jmath} + 3\hat{k}$ prove that $|\underline{A}.\underline{B}|^2 + |\underline{A} \times \underline{B}|^2 = |\underline{A}|^2.|\underline{B}|^2$

Week 14

Topics: Vector differentiation

Pages (99-104)

Del/delta: A vector valued function is denoted and defined as follows:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Gradient: If φ be a scalar valued function then the gradient of φ is denoted by grad φ or $\underline{\nabla}\varphi$ and is defined as follows:

$$\text{grad } \varphi = \vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\varphi = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$$

Question: Suppose $\varphi = xyz$ then find the Gradient of φ at the point (2, 0, 1).

Solution: Given, $\varphi = xyz$

$$\text{So, } \textbf{grad } \varphi = \vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}xyz\,\hat{\imath} + \frac{\partial}{\partial y}xyz\,\hat{\jmath} + \frac{\partial}{\partial z}(xyz)\hat{k}$$

$$= yz.1\,\hat{\imath} + xz.1\,\hat{\jmath} + xy.1\hat{k}$$

$$\vec{\nabla}\varphi = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\hat{k}$$

At the point (2, 0, 1), grad $\varphi = \vec{\nabla}\varphi = 0\cdot 1\,\hat{\imath} + 2\cdot 1\,\hat{\jmath} + 2\cdot 0\hat{k} = 2\,\jmath$

Question: Suppose $\varphi = (xz - y^2)$ then find the Gradient of φ at the point (3, -1, 1).

Solution: $\boldsymbol{grad}\ \varphi = \vec{\nabla}\varphi = \frac{\partial}{\partial x}(xz - y^2)\hat{\imath} + \frac{\partial}{\partial y}(xz - y^2)\hat{\jmath} + \frac{\partial}{\partial z}(xz - y^2)\hat{k}$

$$= (z.1 - 0)\hat{\imath} + (0 - 2y)\hat{\jmath} + (x.1 - 0)\hat{k}$$

$$= z\hat{\imath} - 2y\hat{\jmath} + x\hat{k}$$

At the point (3, -1, 1), grad $\varphi = \underline{\nabla}\varphi = 1\,\hat{\imath} - 2(-1)\hat{\jmath} + 3\hat{k} = \hat{\imath} + 2\,\hat{\jmath} + 3\hat{k}$

Question: Suppose $\varphi = 2xz - yz$ then find the Gradient of φ at the point (2, -1, 0).

$$grad\ \varphi = \vec{\nabla}\varphi = \frac{\partial}{\partial x}(2xz - yz)\,\hat{\imath} + \frac{\partial}{\partial y}(2xz - yz)\,\hat{\jmath} + \frac{\partial}{\partial z}(2xz - yz)\hat{k}$$

$$= (2z.1 - 0)\hat{\imath} + (0 - z.1)\hat{\jmath} + (2x.1 - y.1)\hat{k}$$

$$= 2z\,\hat{\imath} - z\,\hat{\jmath} + (2x - y)\hat{k}$$

At the point (2, -1, 0), grad $\varphi = \vec{\nabla}\varphi = 2 \cdot 0\,\hat{\imath} - 0\,\hat{\jmath} + (2 \cdot 2 + 1)\hat{k} = 5\hat{k}$

Divergence: If $\vec{A} = f\hat{\imath} + g\hat{\jmath} + h\hat{k}$ be a vector valued function then the divergence of $\vec{A}$ is denoted by div $\vec{A}$ or $\vec{\nabla} \cdot \vec{A}$ and is defined as follows:

$$\text{div}\,\underline{A} \text{ or } \vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (f\hat{\imath} + g\hat{\jmath} + h\hat{k}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Question-1: If $\vec{A} = xz\,\hat{\imath} - xy\,\hat{\jmath} + yz\,\hat{k}$ then find the Divergence of $\vec{A}$ at the point (1,-1, 2).

Solution: Given, $\vec{A} = xz\,\hat{\imath} - xy\,\hat{\jmath} + yz\,\hat{k}$

Now, div $\vec{A}$ or $\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (xz\,\hat{\imath} - xy\,\hat{\jmath} + yz\,\hat{k})$

$$= \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz) = z - x + y$$

At the point (1,-1, 2), div $\underline{A}$ or $\vec{\nabla} \cdot \vec{A} = 2 - 1 - 1 = 0$

Question-2: If $\vec{A} = x^2\hat{\imath} - z^2\hat{\jmath} - y^2 z\hat{k}$ then find the Divergence of $\underline{A}$ at the point (-3,0, 2).

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2 z) = 2x - 0 - y^2 = 2x - y^2$$

At the point (-3, 0, 2) ,div $\vec{A}$ or $\vec{\nabla} \cdot \vec{A} = 2(-3) - 0 = -6$

Exercise: Let $\vec{P} = x^2yz\,\hat{\imath} - (xz - y^2)\hat{\jmath} - yz\hat{k}$, find the Divergence of $\vec{P}$ at the point (2, 1, -3).

Curl: If $\vec{A} = f\hat{\imath} + g\hat{\jmath} + h\hat{k}$ be a vector valued function then the curl of $\vec{A}$ is denoted by curl $\vec{A}$ or $\vec{\nabla} \times \vec{A}$ and is defined as follows:

$$\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \hat{\imath}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

Question: If $\vec{A} = x^2\hat{\imath} - z^2 j + y^2\hat{k}$ then find the Curl of $\vec{A}$ at the point (1,0, 1).

Solution: Given, $\vec{A} = x^2\hat{\imath} - z^2 j + y^2\hat{k}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -z^2 & y^2 \end{vmatrix}$$

$= (\frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z} z^2)\,\hat{\imath} - (\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial z} x^2)\,\hat{\jmath} + (-\frac{\partial}{\partial x} z^2 - \frac{\partial}{\partial y} x^2)\hat{k}$

$= (2y + 2z)\hat{\imath} - (0 - 0)\hat{\jmath} + (0 - 0)\hat{k}$

$= (2y + 2z)\imath$

At the point (1,0, 1), $\vec{A}$ or $\vec{\nabla} \times \vec{A} = (2.0 + 2.1)\hat{\imath} = 2\,\imath$

Exercise: If $\vec{A} = xy\hat{\imath} - yz\hat{\jmath} + z^2\hat{k}$ then find the Curl of $\underline{A}$ at the point (-1,1, 2).

Question: Suppose, $\vec{P} = xy\,\hat{\imath} - yz\,\hat{\jmath} + z^2\hat{k}$, $\vec{Q} = y\hat{\imath} + 2x\hat{\jmath} - 3z\hat{k}$, $\vec{R} = x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k}$ and $\varphi = xyz$. Show that (i) $(\vec{P} \cdot \vec{\nabla})\varphi = (\vec{\nabla}\varphi) \cdot \vec{P}$ *and* $(ii)(\vec{R} \cdot \vec{\nabla})\varphi = (\vec{\nabla}\varphi) \cdot \vec{R}$ (***yourself***)

Solution: (i)

Here, $\vec{P} \cdot \vec{\nabla} = (xy\hat{\imath} - yz\hat{\jmath} + z^2\hat{k}).(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}) = xy\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}$

Therefore $(\vec{P} \cdot \vec{\nabla})\varphi = (xy\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z})\,(xyz)$

$$= xy\frac{\partial}{\partial x}(xyz) - yz\frac{\partial}{\partial y}(xyz) + z^2\frac{\partial}{\partial z}(xyz)$$

$$= xy.yz - yz.xz + z^2.xy = xy^2z$$

Again, $\vec{\nabla}\varphi = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k})\,(xyz)$

$= \frac{\partial}{\partial x}(xyz)\,\hat{\imath} + \frac{\partial}{\partial y}(xyz)\,\hat{\jmath} + \frac{\partial}{\partial z}(xyz)\hat{k}$

$= yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$

$$(\vec{\nabla}\varphi) \cdot \vec{P} = (yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}).(xy\hat{\imath} - yz\hat{\jmath} + z^2\hat{k})$$

$= xy^2z - xyz^2 + xyz^2 = xy^2z.$

Thus $(\vec{P} \cdot \vec{\nabla})\varphi = (\vec{\nabla}\varphi) \cdot \vec{P}$ **(Showed)**

Question: Prove (i) $\vec{\nabla} \cdot (\vec{Q} + \vec{R}) = \vec{\nabla} \cdot \vec{Q} + \vec{\nabla} \cdot \vec{R}$ (ii) $\vec{\nabla} \cdot (\vec{P} + \vec{Q}) = \vec{\nabla} \cdot \vec{P} + \vec{\nabla} \cdot \vec{Q}$ $(\boldsymbol{youself})$

Solution: Here $\vec{Q} + \vec{R} = y\hat{\imath} + 2x\hat{\jmath} - 3z\hat{k} + x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k}$

$$= (x^2 + y)\,\hat{\imath} + (2x - z^2)\,\hat{\jmath} - (3z + y^2)\,\hat{k}$$

$$\vec{\nabla} \cdot (\vec{Q} + \vec{R}) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) . \{(x^2 + y)\,\hat{\imath} + (2x - z^2)\,\hat{\jmath} - (3z + y^2)\,\hat{k}\}$$

$$= \frac{\partial}{\partial x}(x^2 + y) + \frac{\partial}{\partial y}(2x - z^2) - \frac{\partial}{\partial z}(3z + y^2) = 2x + 0 - 3 = 2x - 3$$

Again, $\vec{\nabla} \cdot \vec{Q} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(y\hat{\imath} + 2x\hat{\jmath} - 3z\hat{k})$

$$= \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(2x) - \frac{\partial}{\partial z}(3z) = 0 - 0 - 3 = -3$$

And $\vec{\nabla} \cdot \vec{R} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k})$

$$= \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2) = 2x - 0 - 0 = 2x$$

$\vec{\nabla} \cdot \vec{Q} + \vec{\nabla} \cdot \vec{R} = -3 + 2x = 2x - 3.$

Thus $\vec{\nabla} \cdot (\vec{Q} + \vec{R}) = \vec{\nabla} \cdot \vec{Q} + \vec{\nabla} \cdot \vec{R}$ **(Proved)**

Question: (a) If $\varphi = x^2yz$ then prove that $\vec{\nabla} \times (\vec{\nabla}\varphi) = 0.$

(b) If $\varphi = xyz$ and $\underline{A} = xz\hat{\imath} - y^2\hat{\jmath} + z^2\hat{k}$ then show that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0.$

Solution: (a)

$$\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\varphi = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(x^2yz)\,\hat{\imath} + \frac{\partial}{\partial y}(x^2yz)\,\hat{\jmath} + \frac{\partial}{\partial z}(x^2yz)\,\hat{k}$$

$$= 2xyz\,\hat{\imath} + x^2z\,\hat{\jmath} + x^2y\,\hat{k}$$

Now, $\vec{\nabla} \times (\vec{\nabla}\varphi) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix}$

$$= \hat{\imath}\{\frac{\partial}{\partial y}(x^2y) - \frac{\partial}{\partial z}(x^2z)\} - \hat{\jmath}\{\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial z}(2xyz)\} + \hat{k}\{\frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(2xyz)\}$$

$$= (x^2 - x^2)\hat{\imath} - (2xy - 2xy)\hat{\jmath} + (2xz - 2xz)\hat{k} = 0.$$

(b) $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y^2 & z^2 \end{vmatrix}$

$$= \hat{\imath}\,(\frac{\partial}{\partial y}z^2 + \frac{\partial}{\partial z}y^2) - \hat{\jmath}\,(\frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial z}xz) + \hat{k}(-\frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}xz)$$

$$= (0+0)\hat{\imath} - (0 - x.1)\hat{\jmath} + (-0-0)\hat{k} = x\jmath$$

Now, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}).(x\hat{\jmath}) = \frac{\partial}{\partial y}(x) = 0.$

<u>**Question:**</u> If $\varphi = xyz^2$ and $\vec{A} = x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k}$ then show $\vec{\nabla} \cdot (\varphi\vec{A}) = (\vec{\nabla}\varphi) \cdot \vec{A} + \varphi(\vec{\nabla} \cdot \vec{A})$

<u>**Solution:**</u> Here $\varphi\vec{A} = xyz^2.(x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k}) = x^3yz^2\hat{\imath} - xyz^4\hat{\jmath} - xy^3z^2\hat{k}$

So, $\vec{\nabla} \cdot (\varphi\vec{A}) = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}).(x^3yz^2\hat{\imath} - xyz^4\hat{\jmath} - xy^3z^2\hat{k})$

$$= \frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(xyz^4) - \frac{\partial}{\partial z}(xy^3z^2)$$

$$= 3x^2yz^2 - xz^4 - 2xy^3z$$

Also, $\vec{\nabla}\varphi = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}).xyz^2 = \frac{\partial}{\partial x}(xyz^2)\,\hat{\imath} + \frac{\partial}{\partial y}(xyz^2)\,\hat{\jmath} + \frac{\partial}{\partial z}(xyz^2)\hat{k}$

$$= yz^2\,\hat{\imath} + xz^2\,\hat{\jmath} + 2xyz\hat{k}$$

Therefore $(\vec{\nabla}\varphi) \cdot \vec{A} = (yz^2\,\hat{\imath} + xz^2\,\hat{\jmath} + 2xyz\hat{k}) \cdot (x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k}) = x^2yz^2 - xz^4 - 2xy^3z$

Again, $\varphi(\vec{\nabla} \cdot \vec{A}) = xyz^2\,[(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}) \cdot (x^2\hat{\imath} - z^2\hat{\jmath} - y^2\hat{k})]$

$$= xyz^2 \{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\} = 2x^2yz^2$$

$$(\vec{\nabla}\varphi)\cdot\vec{A} + \varphi(\vec{\nabla}\cdot\vec{A}) = 2x^2yz^2 + x^2yz^2 - xz^4 - 2xy^3z = 3x^2yz^2 - xz^4 - 2xy^3z$$

Hence, $\vec{\nabla}\cdot(\varphi\vec{A}) = (\vec{\nabla}\varphi)\cdot\vec{A} + \varphi(\vec{\nabla}\cdot\vec{A})$ **(Showed)**

Week 15

Topics: Vector integration,

Green's theorem Pages (104- 107)

Vector Integration: If $R(u) = r_1(u)\hat{\imath} + r_2(u)\hat{\jmath} + r_3(u)\hat{k}$ then

$$\int R(u)du = \int\{r_1(u)\hat{\imath} + r_2(u)\hat{\jmath} + r_3(u)\hat{k}\}du.$$

$$= \hat{\imath}\int r_1(u)du + \hat{\jmath}\int r_2(u)du + \hat{k}\int r_3(u)du$$

Question: Suppose $R(u) = 3\hat{\imath} + (u^3 + 4u^7)\hat{\jmath} + u\hat{k}$ Find $\int R(u)du$ and $\int_0^2 R(u)du$.

Solution: Given, $R(u) = 3\hat{\imath} + (u^3 + 4u^7)\hat{\jmath} + u\hat{k}$

So, $\int R(u)du = \int\{3\hat{\imath} + (u^3 + 4u^7)\hat{\jmath} + u\hat{k}\}du$

$$= 3u\hat{\imath} + (\frac{u^4}{4} + 4.\frac{u^8}{8})\hat{\jmath} + \frac{u^2}{2}\hat{k} + c = 3u\hat{\imath} + (\frac{u^4}{4} + \frac{u^8}{2})\hat{\jmath} + \frac{u^2}{2}\hat{k} + c$$

Again,

$$\int_0^2 R(u)du = \int_0^2\{3\hat{\imath} + (u^3 + 4u^7)\hat{\jmath} + u\hat{k}\}du$$

$$= \left[3u\hat{\imath} + (\frac{u^4}{4} + \frac{u^8}{2})\hat{\jmath} + \frac{u^2}{2}\hat{k}\right]_0^2$$

$$= 3(2-0)\hat{\imath} + (\frac{2^4}{4} + \frac{2^8}{2} - 0 - 0)\hat{\jmath} + (\frac{2^2}{2} - 0)\hat{k} = 6\hat{\imath} + 132\hat{\jmath} + 2\hat{k}$$

Exercise:

1. Suppose $R(u) = (2u^3 - 6u)\,\hat{\imath} - 3u^5\,\hat{\jmath} - 32u\,\hat{k}$ Find $\int R(u)du$ and $\int_0^3 R(u)du$.

   Ans: $36\,\hat{\imath} - 192\,\hat{k}$

2. Suppose $P(u) = 2u\hat{\imath} + (u - 3u^4)\hat{\jmath} - u^2\hat{k}$ Find $\int P(u)du$ and $\int_{-1}^{2} P(u)du$.

   Ans: $3\,\hat{\imath} - \frac{183}{10}\hat{\jmath} - 3\,\hat{k}$

**** If the position vector of a point is** $\underline{\boldsymbol{r}} = \boldsymbol{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}$ **then** $\boldsymbol{d\underline{r}} = \boldsymbol{dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}}$

Question: If $\vec{F} = 2x\,\hat{\imath} - xy^2\,\hat{\jmath} + yz^3\,\hat{k}$, then find the value of $\int \vec{F}.d\vec{r}$

Solution: Here $\int \vec{F}.d\vec{r} = \int (2x\hat{\imath} - xy^2\hat{\jmath} + yz^3\hat{k}).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$

$$= \int (2x\,dx - xy^2\,dy + yz^3\,dz) = 2\cdot\frac{x^2}{2} - x\cdot\frac{y^3}{3} + y\cdot\frac{z^4}{4} + c$$

$$= x^2 - \frac{xy^3}{3} + \frac{yz^4}{4} + c \quad \text{(Ans:)}$$

Question: If $\vec{F} = xz\hat{\imath} - y^2\hat{\jmath} + mz^5\hat{k}$ and $f = xy^2z$ find the value of $\int \vec{F}.d\vec{r}$ and $\int f.d\vec{r}$

Green's Theorem: Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives on some open set containing R, then

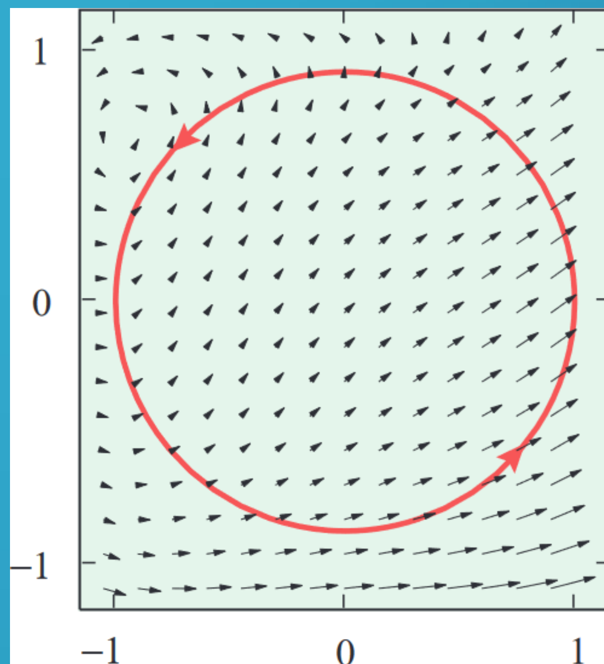
$$\oint_c f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Example-1: Find the work done by the force field $\bar{F}(x,y,z) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction.

Solution:

The work W performed by the field is

$$W = \oint_c \vec{F}\cdot d\vec{r}$$
$$= \oint_c \left(e^x - y^3\right)dx + \left(\cos y + x^3\right)dy$$
$$= \iint_R \left[\frac{\partial}{\partial x}\left(\cos y + x^3\right) - \frac{\partial}{\partial y}\left(e^x - y^3\right)\right]dA$$
$$= \iint_R \left(3x^2 + 3y^2\right)dA$$
$$= 3\iint_R \left(x^2 + y^2\right)dx\,dy$$
$$= 3\int_0^{2\pi}\int_0^1 \left(r^2\right)r\,dr\,d\theta$$
$$= 3\left[\frac{r^4}{4}\right]_0^1 \left[\theta\right]_0^{2\pi} = \frac{3\pi}{2}$$



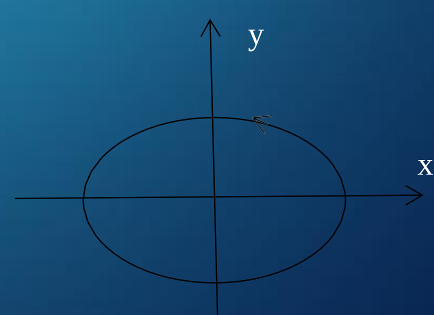
Example-2: Use a line integral to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: The ellipse, with counterclockwise orientation, can be represented parametrically by

$$x = a\cos\theta, \quad y = b\sin\theta, \qquad 0 \le \theta \le 2\pi$$

If we denote this curve by C, then the area A enclosed by the ellipse is

$$A = \frac{1}{2}\oint_c x\,dy - y\,dx$$
$$= \frac{1}{2}\int_0^{2\pi}\left[(-b\sin\theta)(-a\sin\theta) + (a\cos\theta)(b\cos\theta)\right]d\theta$$
$$= \frac{1}{2}ab\int_0^{2\pi}\left(\sin^2\theta + \cos^2\theta\right)d\theta$$
$$= \frac{1}{2}ab\int_0^{2\pi} 1\,d\theta = \pi ab$$



Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

1. $\oint_c \left(x^2 - 2y^2\right)dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$.
2. $\oint_c y\tan^2 x\,dx + \tan x\,dy$, where C is the circle $x^2 + \left(y+1\right)^2 = 1$.
3. $\oint_c \left(x^2 - 3y\right)dx + 3x\,dy$, where C is the circle $x^2 + y^2 = 4$.
4. $\oint_c \left(e^x + y^2\right)dx + \left(e^y + x^2\right)dy$, where C is the boundary of the region between $y = x^2$ & $y = 2x$
5. $\oint_c x^2 y\,dx - y^2 x\,dy$, where C is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle $x^2 + y^2 = 16$.

Week 16 & 17

Topics: Divergence Theorem and , Stokes theorem

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Divergence theorem: Let G be a solid whose surface σ is oriented outward.

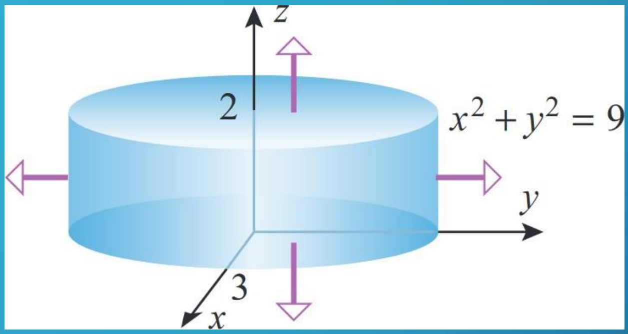
If $\vec{F}\left(x,y,z\right) = f\left(x,y,z\right)\hat{i} + g\left(x,y,z\right)\hat{j} + h\left(x,y,z\right)\hat{k}$ where f , g , and h have continuous first partial derivatives on some open set containing G , and if n is the outward unit normal on σ, then

$$\iint_{\sigma} \vec{F}\cdot\hat{n}\,dS = \iiint_{G} div\,\vec{F}\,dV$$

Example: Use the Divergence Theorem to find the outward flux of the vector field $\vec{F}\left(x,y,z\right) = x^3\hat{i} + y^3\hat{j} + z^2\hat{k}$ across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$

Solution: Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is $div\, F = \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^2\right) = 3x^2 + 3y^2 + 2z$

Therefore, the flux across σ is



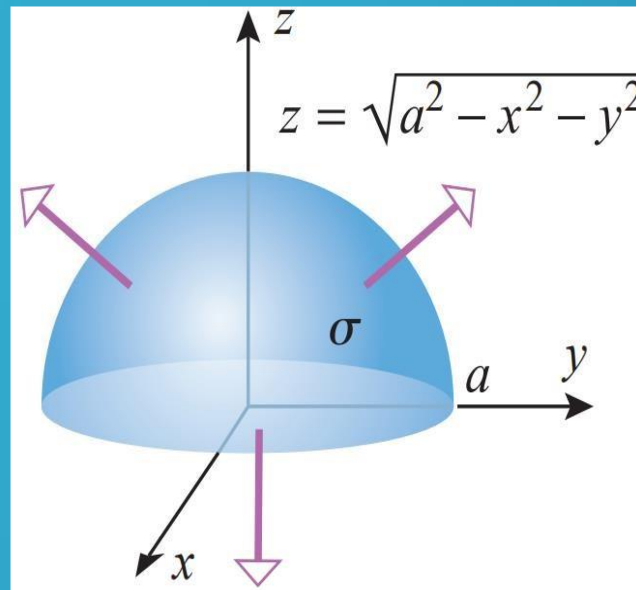
$$\iint_\sigma \vec{F}\cdot\hat{n}\,dS = \iiint_G \left(3x^2 + 3y^2 + 2z\right)dV$$
$$= \int_0^{2\pi}\int_0^3\int_0^2 \left(3r^2 + 2z\right) r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3 \left[3r^3 z + z^2 r\right]_{z=0}^{2} dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3 \left(6r^3 + 4r\right) dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{6\,r^4}{4} + 2r^2\right]_0^3 d\theta$$
$$= \frac{279}{2}\int_0^{2\pi} d\theta = 279\pi$$

Example: Use the Divergence Theorem to find the outward flux of the vector field $\vec{F}\left(x, y, z\right) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ across the surface of the region that is enclosed by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and the plane z = 0

Solution: Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is $div\,\vec{F} = \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right) = 3x^2 + 3y^2 + 3z^2$

Therefore, the flux across σ is

$$\iint_{\sigma} \vec{F}\cdot\hat{n}\,dS = \iiint_{G} \left(3x^2+3y^2+3z^2\right)dV$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} \left(3\rho^2\right)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= 3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= 3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\frac{\rho^5}{5}\right]_{\rho=0}^{a} \sin\phi\, d\phi\, d\theta$$
$$= \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi\, d\theta$$
$$= \frac{3a^5}{5}\int_0^{2\pi} \left[-\cos\phi\right]_0^{\frac{\pi}{2}} d\theta$$
$$= \frac{3a^5}{5}\int_0^{2\pi} d\theta = \frac{6\pi a^5}{5}$$



Use the Divergence Theorem to find the flux of F across the surface σ with outward orientation.

1. $\vec{F}(x,y,z) = z^3\,\hat{i} - x^3\,\hat{j} + y^3\,\hat{k}$, where σ is the sphere $x^2+y^2+z^2=a^2$

2. $F(x,y,z) = (x-z)\hat{i} + (y-x)\hat{j} + (z-y)\hat{k}$, where σ is the surface of the cylindrical solid bounded by $x^2+y^2=a^2$, $z=0$, and $z=1$.

3. $\vec{F}(x,y,z) = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$; σ is the surface of the solid bounded by the paraboloid $z=4-x^2-y^2$ and the xy-plane.

4. $\vec{F}(x,y,z) = x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$; σ is the surface of the cylindrical solid bounded by $x^2+y^2=4$, $z=0$, and $z=3$.

5. $\vec{F}(x,y,z) = (x^3 - e^y)\hat{i} + (y^3 + \sin z)\hat{j} + (z^3 - xy)\hat{k}$, where σ is the surface of the solid bounded above by $z = \sqrt{4 - x^2 - y^2}$ and below by the xy-plane

6. $\vec{F}(x,y,z) = 4xz\,\hat{i} + yz^2\,\hat{j} + z\,\hat{k}$, where σ is the surface of the solid bounded above by $z = \sqrt{a^2 - x^2 - y^2}$ and below by the xy-plane.

Stokes' Theorem: Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation.

If $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$ where f, g, and h are continuous and have continuous first partial derivatives on some open set containing σ, and $\hat{n}$ is the unit normal vector, then $\int_c \vec{F} \cdot d\vec{r} = \iint_\sigma (curl\ \vec{F}) \cdot \hat{n}\, dS$

Example: Verify Stokes' Theorem for the vector field $\vec{F}(x,y,z) = 2z\,\hat{i} + 3x\,\hat{j} + 5y\,\hat{k}$, taking σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation, and C to be the positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy-plane.

Solution: Since σ is oriented up, the positive orientation of C is counterclockwise looking down the positive z-axis. Thus, C can be represented parametrically (with positive orientation) by

$x = 2\cos t, \quad y = 2\sin t, \quad z = 0, \quad 0 \leq t \leq 2\pi$

Therefore,

$$\oint_c \vec{F}\cdot d\vec{r} = \oint_c 2z\,dx + 3x\,dy + 5y\,dz$$

$$= \int_0^{2\pi} \left[0 + (6\cos t)(2\cos t) + 0\right] dt$$

$$= \int_0^{2\pi} 12\cos^2 t\; dt$$

$$= 6\int_0^{2\pi} (1 + \cos 2t)\,dt = 6\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = 12\pi$$

Again, $\quad curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\hat{i} + 2\hat{j} + 3\hat{k}$

Since σ is oriented up and is expressed in the form $z = g(x, y) = 4 - x^2 - y^2$, it follows

$$\iint_\sigma \left(curl\ \vec{F}\right)\cdot\hat{n}\,dS$$

$$= \iint_R \left(curl\ \vec{F}\right)\cdot\left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial x}\hat{j} + \hat{k}\right) dA$$

$$= \iint_R \left(5\hat{i} + 2\hat{j} + 3\hat{k}\right)\cdot\left(2x\hat{i} + 2y\,\hat{j} + \hat{k}\right) dA$$

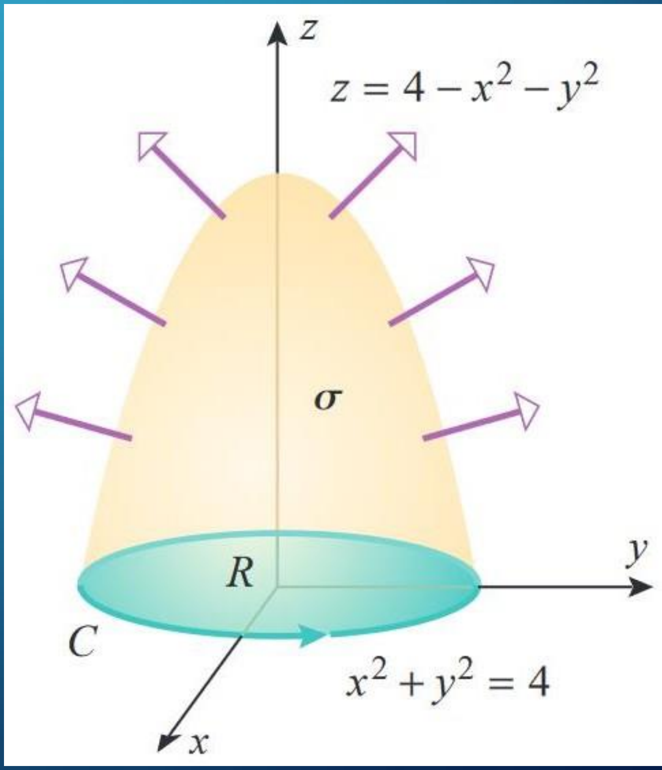
$$= \iint_R (10x + 4y + 3)\,dA$$

$$= \int_0^{2\pi}\int_0^2 (10r\cos\theta + 4r\sin\theta + 3)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[10\frac{r^3}{3}\cos\theta + 4\frac{r^3}{3}\sin\theta + 3\frac{r^2}{2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right) d\theta$$

$$= \left[\frac{80}{3}\sin\theta - \frac{32}{3}\cos\theta + 6\theta\right]_0^{2\pi} = 12\pi$$



Verify Stokes' Theorem by evaluating the line integral and the surface integral. Assume that the surface has an upward orientation.

1. $\vec{F}(x,y,z) = 3z\,\hat{i} + 4x\,\hat{j} + 2y\,\hat{k}$, where σ is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

2. $\vec{F}(x,y,z) = (z-y)\hat{i} + (z+x)\,\hat{j} - (x+y)\hat{k}$; σ is the boundary of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane.